Corresponding Hierarchies and Compatible Measures on Regular Languages of Infinite Words: Memoire de Recherche

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1 Introduction

In the theory of languages of infinite words and the finite state automata capable of recognizing them, the cornerstone was set in 1960 by J.R. Büchi [10]. Studying the decidability of the fragment of second order logic known as Monadic Second Order Logic (MSO1), Büchi proposed a finite state, nondeterministic automaton having a certain criteria by which this machine would accept or reject infinite sequences of symbols, or rather *words* over some finite set. With further investigation, the sets of infinite sequences accepted by a Büchi automaton– *languages*– formed a proper class in the set of all languages of infinite words over a fixed finite alphabet : these are now known as the ω-*Regular* or ω-*Rational* languages (ω-REG).
[11]

The properties of this class became then of primary interest in automata theory, formal language theory, and theoretical computer science, having applications in the latter field especially for the study of nonterminating systems and programming languages. Other authors in these fields such as McNaughton, Muller, Landweber, and Kurt Wagner ([8], [9], [7], [11]) soon expanded upon Büchi's work. Of particular interest were the proposed variants of Büchi's initial automaton construction, such as more complex or restrictive acceptance conditions, or requiring an automaton be deterministic or not.

It was observed that different modes of acceptance could directly influenced the class of languages an automaton could accept, and this pattern then raised interest as to the connection between other properties of a class of languages (for example, the topology complexity) and the complexity and/or determinancy of an automaton's acceptance criterion.

It was found that some modes of acceptance were strictly weaker than others, and so defined proper subclasses of ω-REG, although many possible modes recognizing the entire class of ω-regular languages have been shown to be equivalent. In particular, it is a well known result that the class of languages accepted by deterministic Büchi automata defines a strictly smaller class of languages than accepted by nondeterministic Büchi– the latter class is equivalent to the entire class ω-REG. Furthermore, we have that nondeterministic *and* determinstic Rabin, Streett, Muller, and parity automata also accept exactly ω-REG ; [4] covers the definitions of these other automata and the associated transformations proving their equivalency.

A paper by Landweber [7] looked at six subclasses of ω-REG as defined by an automaton's mode of acceptance. Viewing the space of infinite words over a finite alphabet as homeomorphic to the space of infinite binary sequences (2^ω), a reasonable approach to characterize these classes was to employ classical methods from descriptive set theory so to analyze the topological complexity of the ω languages. This turned out to be quite fruitful : there exist direct correspondences between automata-theoretic notions of acceptance and low classes of the Borel hierarchy. For example, the proper subclass of languages accepted by deterministic Büchi machines coincided with the Π^0_2 or G_δ ω-regular languages, and that the boolean closure of this class was exactly the class ω-REG. In other words, there is a tight correlation between ascension in the Borel hierarchy on $\mathcal{P}(\Sigma^\omega) \bigcap \omega-$REG and the act of increasing the complexity of acceptance conditions for an automata.

The intersection between automata and descriptive set theory is more readily observable by an example :
In descriptive set theory, for a finite set A of natural numbers, we consider the usual product of discrete topologies on the space A^ω of infinite sequences of elements from A. An open set for this topology, say $U \subseteq A^\omega$, is characterized by having a definition using only existential quantification : $x \in U \Leftrightarrow \exists i \in \mathbb{N}$ such that $x[i] \in U$, where $x[i]$ denotes the initial segment of length i of the infinite sequence $x \in A^\omega$.
Switching frames of reference, we might characterize an open ω-regular *language* $L \subseteq A^\omega$ over finite alphabet A by way of a finite state automaton $\mathcal{A}$ which recognizes or accepts a word x if and only if there *exists* a natural number i such that the prefix $x[i] \in A^i$ belongs to the finitary language V [1] over alphabet A. The language L thus admits the expression $L = V.A^\omega$; any word of L is an infinite length word having some prefix in V.

1. requiring also V be calculable by a finite state machine on finite words

In light of these correspondences, the study of the ω- regular languages gained traction by using classical results and techniques from descriptive set theory, such as the notion of Wadge reducibility and difference hierarchies of the ambiguous Borel classes (Δ^0_α for $\alpha > 1$ a countable ordinal).

In this paper we intend to present some of these findings concerning the various hierarchies with which one can classify the ω-Regular languages. We look beyond the descriptive set theory measures of complexity and introduce the *Wagner hierarchy*, induced by the *chain* and *superchain* measures, and are shown in [11] to be invariant of the specific automaton used to recognize a language. The proof we present from [11] in section 3 of this paper demonstrates this invariance property.
The other pertinence of studying Wagner's measures is that the resulting hierarchy is a proper refinement of the Borel classes of $\mathcal{P}(\Sigma^\omega)$ occupied by the ω-regular languages.

After looking at the so called *exact* complexity classes, we will give another measure Wagner proposes in [11] which has an easy correspondance with the chain measures. This other measure, called the *Rabin Index*, is different than the chain measure in that it does not enjoy the same sort of invariant properties, and instead looks at properties of one type of automaton– a *Rabin automaton*, defined by Michael Rabin in his 1969 paper looking at the decidability of second order theories. We look at the complexity measure, the *Rabin Index*, on these machines to highlight alternating nature exhibited by the ω-regular languages.

If (as in the case of the chain measure but not necessarily the Rabin index) this family of conditions is also descending and linearly ordered by set inclusion, the language defined can thus be expressed as a union of *differences*. This is to say we can write a language L as

$$L = (L_1 - L_2) \cup L_3 - L_4 \cup ...L_{n-1} \cup L_n \text{ where } L_1 \supseteq L_2 \supseteq ... \supseteq L_{n-1} \supseteq L_n$$

Visually, L can be pictured as the shaded parts of the target as below.

These observations will lead us naturally to our study of difference hierarchies.

In 1992, work by Rana Barua appropriates the classical result of Hausdorff and Kuratowski about the Δ^0_α difference hierarchy, defining an automata-based hierarchy on ω-REG : roughly speaking, we place an ω-regular language in level n of Barua's hierarchy if it is the union of pairwise differences of finite, nested, decreasing sequence of n languages, each recognizable by *deterministic* Büchi automata$\star$.
We will present the proof that any $L \in \omega$-REG can be placed in this hierarchy ; indeed, given such an L, it is decidable at which n L resides. [1] also asserts the converse, that if a language $L \subseteq \Sigma^\omega$ can be expressed as $\star$, then L is ω-regular.

The other key result of [1] is that the levels of this hierarchy corresponds exactly to the levels of the Hausdorff-Kuratowski difference hierarchy, whence the latter is restricted to just those subsets of Σ^ω that are ω-regular. We will present this proof as well.

In section 1 we give preliminaries and notations central to formal language theory to automata theory. A more thorough presentation can be found in [4].
After establishing finite state automata and how they interact with languages via various *modes of acceptance*, we give some characterizations ω-REG and certain of its subclasses. In particular we give a table ((REF TABLE)) relating the topological interpretations and set theoretic representations of classes of languages in correspondence with varying modes of finite state automaton acceptance. For this we will include in section 1 some basic topology and descriptive set theory.

In want of a finer complexity measure on ω-languages than those provided by descriptive set theory, in section 2 we introduce Wagner's chain and superchain measures, and the resulting hierarchy induced on ω-REG (referred to as the *Wagner hierarchy*) consisting of the "exact" complexity classes C^n_m, D^n_m, E^n_m, $m, n \geq 1$. Next we present a coarser variant of these classes, the *downwards* classes $\widehat{C}^n_m, \widehat{D}^n_m, \widehat{E}^n_m$ and show their equivalencies with certain Borel classes [2]. Section 2 also gives Wagner's [11] proof of the disjointedness of the exact complexity classes, and thus the Wagner hierarchy is a proper refinement of the ω-regular portion of the Borel hierarchy. Moreover, the way this is proved also illustrates the *invariant property* of the chain and superchain measures :

2. we should specify here the equivalency is only once we have restricted the Borel sets to those which are ω-regular ; this will be discussed in detail later on.

any two finite state automaton recognizing the same language will always be in the same complexity class.
We then define another measure on complexity from [11] which is dependent upon the specific automaton accepting the language : the *Rabin Index.* It is a measure on the complexity of acceptance conditions for a **Rabin automaton** (defined in section 1), and it is still compatible with the chain and superchain measures. We give the results about the correspondance between the two. Section 2 will end with some decidability results.

In section 3 we once again we switch our attention to a topological standpoint to illustrate how classical complexity measures in descriptive set theory have applications in the study of ω-REG. In particular, we define the *Hausdorff-Kuratowski Difference Hierarchy* (abbreviated sometimes HKDC) on the Borel classes $\Delta_{\alpha+1}$ of Polish spaces [3] for α a countable non-null ordinal.
We then present an analagous difference hierarchy defined in [1] and give Barua's proof that any ω-regular language sits in some level of that hierarchy.

At the time of Barua's work, it was already well known from [7] that the ω-regular subsets of Σ^ω were exactly the class of Δ^0_3 subsets of this same space. Immediately then they admit some level in the HKDC, taking the case of the Borel class Δ^0_3, giving first a weak correspondance of these two hierarchies. But the (countably many) ω-regular languages only stratifying an initial segment of the Borel hierarchy, it is not clear that there should be a tighter correspondence. Indeed, this is the main result of Barua [1] : that any ω-regular L will admit the same rank n in Barua's hierarchy as it would in the HKDC. Section 4 will present Barua's proof of the seperation-like theorem from which the result follows almost immediately.
The techniques we will see employed are generalizations of certain ways in which Landweber [7] characterized the ω-regular G_δ subsets. We present Barua's result here, emphasizing the separation theorem central to the proof, and show how it is a generalization of a certain characterization of language computable by deterministic Büchi automata : that it be accepted also by a *full-table* Muller automata. We then revisit Wagner's chain measure and see how this property manifests itself in the class $\widehat{C^1_2}$ by giving a more thorough illustration of calculating the measures $m+^+, m^-, n^+, n^-$ for languages accepted by full-table Muller automata.

It is important also to see Barua's method of proof because it gives an algorithm for computing an ω-regular language's position in the restricted HKDC (equivalently, Barua's hierarchy). Moreover this algorithm suggests yet another way of classifying ω-REG, namely by way of the complexity of Muller automaton. We define the classes of this hierarchy as well. Section 4 will end with decidability results. At the end we will briefly discuss why the results and methods of [11], [1], [7] are important to the study of languages of finite words and how they might be extended to other studies. We will include references to recent surveys and studies in connection with the topics of this paper.

2 Preliminaries

2.1 Notations ; Basic Definitions of Formal Language Theory

We use the ordinal ω to denote the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$.
An **alphabet** Σ is a finite set of symbols (letters). We mostly will use a, b as our letters for working purposes.
Finite words over Σ are finite sequences $x = (a_1, a_2, ..., a_n)$ of letters of Σ ; i.e. for all $1 \leq i \leq n < \omega$, $a_i \in \Sigma$.
Finite words will typically be denoted by lowercase greek letters $x, y, z....$
The **length** of word x is denoted $|x|$, and similarly for cardinality of sets A we write $|A|$.
The unique word of length 0 is the **empty word** and is denoted by the symbol ε.
The ith **coordinate** of $x = (a_1, ..., a_n)$ is written $x(i)$; it is a singleton of Σ. $x(0) = \varepsilon$. The **concatenation** of letters $a, b \in \Sigma$ is $a.b$, and for words x, y finite words over the same alphabet Σ their concatenation is $x.y := (x(1).x(2).....x(|x|).y(1).....y(|y|))$.

For words x, y such that $|x| \leq |y|$, x is a **prefix** of y if the sequence x is an initial segment of the sequence y, i.e. if $x(i) = y(i)$ for all $1 \leq i \leq |x|$. This relation is denoted by $x \sqsubseteq y$.
A **language** finite words over Σ is a set whose elements are finite words. Σ^* denotes the set of all languages of finite words over Σ. Its complement, the set $\Sigma^* \setminus L$ is written L^C. Languages in Σ^* will typically be represented by uppercase latin letters $L, K, U, ...$
The concatenation operation is extended to languages : for $L, K \subseteq \Sigma^*$, their concatenation is the language

$$L.K := \{x.y \mid x \in L, \quad y \in K\}$$

We also define the union $(L \cup K)$ and intersection $(L \cap K)$ of languages L, K to be, respectively,

$$L \cup K := \{x \mid x \in L \quad \vee \quad x \in K\} \tag{1}$$

$$L \cap K := \{x \mid x \in L \quad \wedge \quad x \in K\} \tag{2}$$

3. Polish spaces are topological spaces which are separable (contain a countable dense subset) and admit a compatible, complete metric ; the Borel sets of these spaces are those of the σ-algebra generated by open sets

An **infinite word** (or ω-**word**) α over Σ is an infinite sequence of symbols from Σ, and as for finite words we write $\alpha = (\alpha(1).\alpha(2).....\alpha(n)....)$ where $\alpha(i) \in \Sigma$ for all $i < \omega$. Σ^ω is the set of all infinite words over Σ.
The prefix of the ω-word α is denoted $\alpha[n] = \alpha(0).\alpha(1).....\alpha(n)$ for $n < |\alpha|$. The concatenation of a finite word x and infinite word α is noted $x.\alpha$

We use exponential notation to denote powers of finite languages : for $L \subseteq \Sigma^*$,
— $L^0 = \{\varepsilon\}$
— $L^n = \{a_1 a_2 ... a_n \mid a_i \in L \text{ for all } 1 \leq i \leq n\}$
— $L^* := \bigcup_{n<\omega} L^n$

The last operation is known as the **Kleene Star** of L, and is a language in Σ^*. The ω-**power** of $L \subseteq \Sigma^*$ is defined similarly :
$L^\omega = \{\alpha \in \Sigma^\omega \mid \alpha = (x_1.x_2....x_i...), x_i \in L \forall i < \omega\}$

Definition 1. *For a language L of finite words, the* ***Eilenberg limit*** *of L is the infinite language $\overrightarrow{L}$ defined by*

$$\overrightarrow{L} := \{\alpha \in \Sigma^\omega \mid \alpha[i] \in L \textit{ for infinitely many } i < \omega\} \tag{3}$$

I.e. $\overrightarrow{L}$ contains those ω-words having infinitely many prefixes in L.

2.2 Finite State Automata

Before working with automata on infinite words, it is helpful to introduce what entails the definition of an automaton on finite words.

Definition 2. *A* ***finite state automaton on finite words*** *$\mathcal{A}$ is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where :*
— *Q is a finite set of states*
— *Σ an alphabet*
— *$\delta : Q \times \Sigma \mapsto 2^Q$ the* ***transition relation***
— *$q_0 \in Q$ the initial state*
— *$F \in \mathcal{P}(Q)$ is the system of final states, or* ***acceptance component***

If it holds that $|\delta(q, a)| \leq 1$ for all $q \in Q$ and all $a \in \Sigma$, then $\mathcal{A}$ is a **deterministic** automaton. Else, we say $\mathcal{A}$ is **nondeterministic**.

Automata admit a natural graph theoretic interpretation : consider the directed graph $G_\mathcal{A} = (V, E)$ with the vertices $V = Q$, and the edges being the set $E = \{(q, q') \mid \exists a \in \Sigma \text{ s.t. } \delta(q, a) = q'\}$. The edges are labeled by the letter of Σ which $\mathcal{A}$ reads in transitioning from one state to the next.
The languages **recognized** or **accepted** by $\mathcal{A}$ is denoted $L(\mathcal{A})$.
It is the set

$$\{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$$

Where $\mathcal{A}$ accepts $w \in \Sigma^*$ if and only if there exists a sequence of states (equivalently, a *path* through $G_\mathcal{A}$) $(q_0, q_1, ..., q_n) \in Q^n$ satisfying the following conditions :

1. q_0 is the initial state
2. $(q_i, w(i), q_{i+1}) \in \delta$ for all $0 \leq i \leq n$
3. $q_n \in F$

We let REG denote the class of languages $L \subseteq \Sigma^*$ such that there exists an automate $\mathcal{A}$ with $L = L(\mathcal{A})$, referred to as the **rational** or **regular** languages (over finite words). It is closed under union, concatenation, complementation, and the Kleene-star operation.
Note that the acceptance component of a finite state automaton on finite words is always a subset $F \subseteq Q$, and there is only one notion of acceptance.

In regards to infinite words (or ω-words), the notion of acceptance becomes more complicated : which infinite paths through a finite graph should be considered accepting ? First we define in terms of automata and languages these infinite paths, or *runs*. Runs will typically be denoted by lowercase greek letters $\sigma, \gamma, ...$

Definition 3. *A* ***run*** *of a finite state automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ on a word $\alpha \in \Sigma^\omega$ is an infinite sequence $(q_0, q_1, ..., q_n, ...)$ of states $q_i \in Q$, $i < \omega$, such that (1) and (2) above hold.*

Two important sets are associated with a run : $\mathrm{Occ}(\sigma)$ and $\mathrm{Inf}(\sigma)$, the set of states occuring at least once and the set of states occuring infinitely often, respectively, in the sequence σ.

$$\mathrm{Occ}(\sigma) := \{q \in Q \mid \exists i \in \omega \text{ s.t. } \sigma(i) = q\}$$

$$\mathrm{In}(\sigma) := \{q \in Q \mid \forall i \exists j > i[\sigma(j) = q]\}$$

A run σ can be either **accepting** or **rejecting**, depending on the **mode of acceptance** of the particular automaton reading ω words.
Büchi proposed the first notion of an accepting run in 1960 [2]. Formally,

Definition 4. *A* ***Büchi automata*** *is a finite state automata $\mathcal{A}_B$ as in Definition 1 ; note that the acceptance component is still a subset of states $F \subseteq Q$.*

For a word $\alpha \in \Sigma^\omega$, $\mathcal{A}_B$ accepts α if and only if there exists at least one run σ of $\mathcal{A}_B$ on α such that :

$$\mathrm{In}(\sigma) \cap F \neq \emptyset \tag{4}$$

In other words, ***Büchi acceptance*** *is the condition that on reading input α, the Büchi automaton $\mathcal{A}$ passes infinitely many times through at least one of the states $q \in F$.*

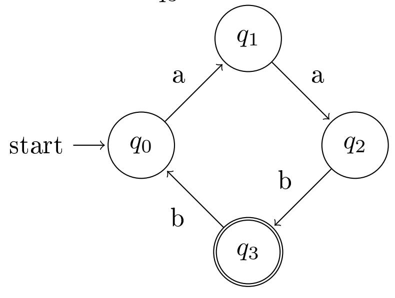
As a first characterization result, we denote by ω-REG the class of **ω-regular languages** we have :

$$\omega - \mathrm{REG} = \{L \subseteq \Sigma^\omega \mid \exists \text{ a nondeterministic Büchi automaton } \mathcal{A}; \quad L = L(\mathcal{A})\} \tag{5}$$

The ω-regular languages are closed by complement and finite union.

Example

Let $\Sigma = \{a, b\}$, and let L be the language of Σ^ω defined by $L = \{\alpha \in \Sigma^\omega \mid \alpha = (aabb)^\omega\}$.
L is recognized by the Büchi automata $\mathcal{A} = (\{q_0, q_1, q_2, q_3\}, \Sigma, \delta, q_0, \{q_3\})$ pictured below. The double outline around state q_3 indicates this is a final state.



Büchi proved in [10] that the ω-regular languages are precisely those which can be expressed as

$$L = \bigcup_{1 \leq i \leq n} U_i . V_i^\omega \qquad \text{where } U_i, V_i \in \mathrm{REG} \quad \forall 1 \leq i \leq n \tag{6}$$

The above collection is called the ω-**Kleene Closure** of the class REG. Another way of phrasing this fact is that any ω-regular language must contain an ultimately periodic word (In this way, checking whether a regular ω-language is empty is decidable).

The reason behind (6) is this : for any word α in ω-regular $L = L(\mathcal{A})$ for Büchi $\mathcal{A}$, there is an accepting run σ and a final state $q_F \in F \subseteq Q$ such that $\sigma(i) = q_F$ for infinitely many i, and in particular there is a word $x \in \Sigma^*$ such that $\delta(q_0, x) = q_F$. Consider now $\mathcal{A}$ as an automaton on *finite* words, x is an accepted word and more generally the set $U = \{x \in \Sigma^* \mid \delta(q_0, x) \in F\}$ is a regular language. Similarly the set $V = \{x \in \Sigma^* \mid \delta(q_F, x) = q_F$ for some $q_F \in F\}$ is a regular language. Taking the union of these two sets over F (where say $|F| = n$) we get the expression in (6).

Three years after Büchi's seminal work, Muller [9] gave a different acceptance condition, equivalent in expressive power to the Büchi machines, yet strengthened the characterization of ω-REG in (5) in that the automaton he defined could be equivalently made deterministic. McNaughton formally proved the equivalency between nondeterministic Büchi and deterministic Muller in 1966 in [8]. The sharper characterization of the class ω-REG motivated other authors to define acceptance conditions, focusing on certain semantics or properties of ω-languages they wanted to capture. Thus ensued an array of alternative modes of acceptance posed by authors such as Streett, Rabin, Landweber, Staiger, etc.. The plurality of definitions invites the questions : how do these automata differ in their expressive power ? Are the different modes of acceptance indicative of structural properties of the language ?

Indeed, we will see that in many cases the complexity of a mode of acceptance strongly influential of the topological and set theoretic properties of the class of languages the condition defines.

We now look at some of these different modes of acceptance.

Definition 5. *A* ***Muller automaton*** $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ *is a finite state automaton equipped with the acceptance component* $\mathcal{F} \subseteq \mathcal{P}(Q)$*, and the following acceptance condition (**Muller acceptance**) :*

$$\text{In}(\sigma) \in \mathcal{F} \tag{7}$$

Theorem 1. *(McNaughton)*
A set $L \subseteq \Sigma^\omega$ *is in* ω*-REG if and only if there exists a Muller automaton* $\mathcal{A}$ *with* $L = L(\mathcal{A})$*.*

We reiterate that in the above theorem, $\mathcal{A}$ can be taken to be deterministic or nondeterministic without any loss of expressive power.

This leads us to another important result we have been hinting at : that although any Büchi machine can be transformed into an equivalent Muller automaton, the latter atuomata cannot always be transformed into a deterministic Büchi automaton accepting the same language.
Therefore,

Theorem 2. *There exist* ω*-regular languages* L *such that* $L \neq L(\mathcal{A})$ *for any deterministic Büchi automaton* $\mathcal{A}$*.*

Proof. Consider the language L over the alphabet $\Sigma = \{a, b\}$ defined as
$L = \{\alpha \in \Sigma^\omega \mid \exists$ only a finite number of occurences of the letter b in $\alpha\}$.
First we note that L is indeed an ω-regular language, for it is accepted by deterministic Muller automaton in Figure 1, with equipped with acceptance component $\mathcal{F} = \{\{q_a\}\}$.

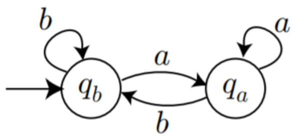


FIGURE 1 – An automaton on infinite words over alphabet $\{a, b\}$.

Theorem 1 also implies the existence of a nondeterministic Büchi machine accepting the same language; explicitly, we can see the automaton in Figure 2 accepts L when we take acceptance component $F = \{\{q_1\}\}$. However $\delta(q_0, a) = \{q_0, q_1\}$ and so the automaton below is nondeterministic.

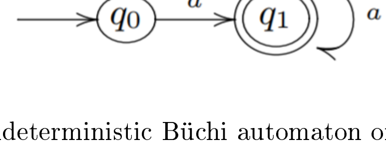


FIGURE 2 – A nondeterministic Büchi automaton on infinite words

Suppose, for contradiction, there exists a deterministic Büchi $\mathcal{A}_B = (Q, \Sigma, \delta, q_0, F)$ recognizing L. Since the ω-word $a^\omega \in L$, there is an accepting run σ by $\mathcal{A}_B$ and some $n_1 < \omega$ such that $\sigma_1 = (q_0, ..., q_{n_1}, ...)$, and $q_{n_1} \in F$. The word $a^n.b.a^\omega \in L$, so it is accepted also by $\mathcal{A}_B$, and as before we find q_{n_2} $\in$Inf$(\sigma_2)\cap F$ where σ_2 is the accepting run in reading $a^{n_1}.b.a^\omega$.
Continuing this reasoning, we can construct an accepting run on the ω-word $\beta = a^{n_1}.b.a^{n_2}.b.....b.a^{n_k}.b....)$ for all $k < \omega$, i.e. β has infinitely many blocks $a^{n_i}.b$, which is a contradiction. $\square$

The above theorem offers an example of two different acceptance conditions having different expressive power. We present the proof to illustrate the unavoidable limitations of deterministic Büchi, since this class is of particular importance to ω-REG as we will see later.
Nonetheless, there exist other finite state automata having exactly the same expressive power as deterministic Muller and therefore recognize the class ω-REG :

Theorem 3. *The following automata are equivalent in expressive power :*
— *deterministic Muller, Rabin, Streett, parity ;*
— *nondeterministic Büchi, Muller, Rabin, Streett, parity.*

We refer to [4] for the definitions of Streett and parity automata and the proofs of the equivalencies. Such proofs are generally algorithms to transform one type of machine into another.

The following definition, concerning a special type of Muller automaton, will be needed. In fact, this paper show how this notion reappears in terms of Wagner's chain measures as well as its importance to results in Barua [1].

Definition 6. *A Muller automaton $\mathcal{A}_M$ is said to be **full-table** if for every $F \in \mathcal{F}$, if there exists a superset $F' \supseteq F$, then we must also have $F' \in \mathcal{F}$.*

The last automaton on ω-words we introduce is particularly illustrating of the hierarchies and complexity measures discussed in this paper.

Definition 7. *A **Rabin automaton** $\mathcal{A}_R$ is a finite state automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ where Q, Σ, δ, q_0 are as defined previously, and additionally $\mathcal{A}_R$ is equipped with the acceptance component $\Omega = \{(E_1, F_1), ..., (E_k, F_k)\}$ for some $k < \omega$, where $E_i, F_i \subseteq Q$ for all $i \leq k$.*
***Rabin acceptance** is the condition that $\mathcal{A}_R$ accepts a word α if and only if there exists a natural number $i < \omega$ and a run σ on α passing infinitely many times through the sets F_i and only finitely many times through E_i. Symbolically,*

$$\mathrm{In}(\sigma) \cap E_i = \emptyset \qquad \wedge \qquad \mathrm{In}(\sigma) \cap F_i \neq \emptyset \tag{8}$$

In section 3, we will see how the chain and superchain measures are compatible with another complexity measure specific to Rabin automata (called the *Rabin Index*) that counts the minimal number of pairs (E_i, F_i) needed to recognize a language $L \in \omega$-REG.

2.3 Topology and Descriptive Set Theory

We give here some basic background and definitions which will be used in later sections. Kechris's text [6] is a useful resource if the reader is less familiar with topology and descriptive set theory.

Equipping the set Σ the discrete topology, we naturally then view Σ^ω with induced product topology. A compatible complete metric is given by d, where for α,$\beta \in \Sigma^\omega$,

$$d(\alpha, \beta) = \frac{1}{2^n} \text{ if } \alpha \neq \beta \text{ and } n = \min(i \in \omega \mid \alpha[i] \neq \beta[i]) \quad \text{and} d(\alpha, \beta) = 0 \text{ if } \alpha = \beta$$

The space Σ^ω is countable, separable, and admits a basis of clopen sets : we fix a basis for τ by defining the sets

$$N_w := \{\alpha \in \Sigma^\omega \mid w \in \Sigma^*,\ w \sqsubseteq \alpha\} \tag{9}$$

The topology thus generated is often known as the *Cantor topology*, but with a more automata-theoretic flavor it is sometimes called the *prefix topology*. (Σ^ω, τ) is Polish and homeomorphic to the Cantor space 2^ω in which the topology is defined analogously for $\Sigma = \{0, 1\}$.

The *Borel σ-algebra* on a topological space (X, τ) is the smallest family of subsets of X containing the open sets and closed by countable union, countable intersection, and complement. The *Borel hierarchy* is the classification into these sets based on their level of generation in the Borel σ-algebra. That is, for α a countable ordinal we define by transfinite recursion the following pointclasses :

$$\mathbf{\Sigma}^0_1 := \{U \subseteq X \mid U \text{ is open in } X\}$$

$$\mathbf{\Pi}^0_1 := \{F \subseteq X \mid F \text{ is closed in } X\} = \neg\mathbf{\Sigma}^0_1$$

$$\mathbf{\Delta}^0_1 := \mathbf{\Pi}^0_1 \cap \mathbf{\Sigma}^0_1$$

$$\begin{array}{cccccccccccccc}
 & & \mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 & \cdots & & & \mathbf{\Sigma}^0_\alpha & & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & & & \nearrow & & \searrow & & \\
\mathbf{\Delta}^0_1 & & & & \mathbf{\Delta}^0_2 & & & \cdots & \mathbf{\Delta}^0_\alpha & & & & \mathbf{\Delta}^0_{\alpha+1} & \cdots \\
 & \searrow & & \nearrow & & \searrow & & & & \searrow & & \nearrow & & \\
 & & \mathbf{\Pi}^0_1 & & & & \mathbf{\Pi}^0_2 & \cdots & & & \mathbf{\Pi}^0_\alpha & & & \cdots
\end{array}$$

For $2 \leq \alpha \leq \omega_1$,

$$\mathbf{\Sigma}^0_\alpha := \{U \subseteq X \mid U = \bigcup_{n<\omega} V_n \text{ where } V_n \in \bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma \quad \forall n < \omega\}$$

$$\mathbf{\Pi}^0_\alpha := \{F \subseteq X \mid F = \bigcap_{n<\omega} U_n \text{ where } U_n \in \bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma \quad \forall n < \omega\}$$

$$\mathbf{\Delta}^0_\alpha := \mathbf{\Pi}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha$$

We note the following properties :
— All classes are closed under finite unions and finite intersections.
— For all $1 \leq \alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha$ and ($\mathbf{\Pi}^0_\alpha$) are closed under countable union and intersection, respectively.
— The class $\mathbf{\Delta}^0_\alpha$ is closed under complement
— For all $1 \leq \alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha = \neg\mathbf{\Pi}^0_\alpha$

The next theorem, presented first in [2] and in [7], establishes where the class of ω-regular languages fits into the Borel σ-algebra. It is credited to Büchi and Landweber [2], and later Trachtenbrot (cf. [11]).

It is necessary to highlight that as there are $2^{\aleph_0}$ Borelian subsets of Σ^ω and only a countable number of ω-regular languages, the class ω-REG stratifies only a small fragment of the Borel hierarchy. Therefore we need to specify when we are considering this smaller class of Borel sets.

We will use ${\Omega^0_\alpha}^{\mathcal{R}}$ denote the pointclass $\Omega \in \{\Sigma, \Pi, \Delta\}$ whence restricted to the class ω-REG, i.e. :

$${\Omega^0_\alpha}^{\mathcal{R}} := \Omega^0_\alpha \cap \{L \subseteq \Sigma^\omega \mid L \in \omega - \text{REG}\}$$

Theorem 4. *Let L be a language of infinite words over a finite alphabet Σ. If L is an ω-regular language, then L is a Δ^0_3 subset of Σ^ω.*

We have another characterization result, suggesting the importance of properties of the G_δ ω-regular languages to the entire class ω-REG.

Corollary 1. *Let $\mathfrak{B}(\Omega)$ denote the Boolean Closure of a pointclass Ω. Then*

$$\omega\text{–REG} = \mathfrak{B}({\Pi^0_2}^{\mathcal{R}}) = \mathfrak{B}({\Sigma^0_2}^{\mathcal{R}})$$

As mentioned in the introduction, the ω-regular languages admit have very nice topological interpretations : for every Borel class of ω-regular languages, one can give a definition of this class in terms of the mode of acceptance necessary to recognize languages of this class.

The table below summarizes the relevant results, linking the Borel complexity of the regular languages to modes of acceptance of a deterministic finite state automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \bullet)$. It is easily observable for the open and closed languages how the complexity quantifiers of an acceptance condition coincides with the classical notion of definability of Borel sets in these classes. These results are proved in Landweber [7].

Borel Class	Acceptance Component	Acceptance Condition
Open (${\Sigma^0_1}^{\mathcal{R}}$)	$F \subseteq Q$	$\forall i(\delta(q_0, \alpha[i]) \in F$
Closed (${\Pi^0_1}^{\mathcal{R}}$)	$F \subseteq Q$	$\exists i(\delta(q_0, \alpha[i]) \in F$
G_δ (${\Pi^0_2}^{\mathcal{R}}$)	$F \subseteq Q$	$\exists$ a run σ on α, $\text{In}(\sigma) \cap F \neq \emptyset$
F_σ (${\Sigma^0_2}^{\mathcal{R}}$)	$\mathcal{F} \subseteq \mathcal{P}(Q)$	$\exists$ a run σ on α and $\exists F \in \mathcal{F}(\text{In}(\alpha) \subseteq F)$
ω-Regular (${\Delta^0_3}^{\mathcal{R}}$)	$\mathcal{F} \subseteq \mathcal{P}(Q)$	$\exists$ a run σ on α and $\exists F \in \mathcal{F}(\text{In}(\alpha) = F)$

Note that the above table gives representations with respect to the Cantor topology. There exist however other topologies one can put on the space of languages of infinite words, many have which been recently proved to be polish.

For example, the *Büchi topology* τ_B defines the open sets of Σ^ω to be the ω-regular sets– i.e. the sets in Δ_3^0 for the Cantor topology. Since this topology is polish, the study of ω-languages classified with respect to the Büchi topology can potentially be enriched by classical descriptive set theory, yet may provide different automata-theory conclusions since the levels of the two resulting hierarchies do not coincide. These results are done in [3].

3 The Wagner Hierarchy

By 1979, the works of Büchi, Hartmanis and Stearns, Landweber, Trachenbrot and Bardsin, Hossley, Wagner, Staiger, and McNaughton (see [10], [5], [7], [12], [8]) effectively completed the study of finite state automata and their topological connection to certain fragments of the Borel hierarchy on the space Σ^ω, the main results of which we just saw.

Naturally, then, the study of the ω-regular languages looked for other possible measures by which to classify these sets. To this end, K. Wagner introduced the measures m^+, m^-, the *chain measures* and n^+, n^-, the *superchain measures*. Roughly, the role of these measures is to consider the deterministic Muller automaton recognizing some $L \in \omega-$REG, and to look (respectively) at the maximal length of alternating sequences (the *chains*) of sets of states, and then the maximal length of chains of chains (*superchains*).The alternating character of chains is defined by if a set of states is considered accepting (positive) or rejecting (negative), relative to the acceptance component $\mathcal{F}$.

An important feature of these measures is that they are proved in [11] to be invariant of the specific automata accepting the language– if $\mathcal{A}$ and $\mathcal{B}$ recognize the same ω-regular language L, the maximal chains and superchains of either automaton must be the same numbers m and n. Defining then an order $\leq$ on ω-REG based on these measures, we obtain the resulting equivalence classes which Wagner names the *exact complexity classes*, denoted C_m^n, D_m^n, and E_m^n. We will see that the entire class ω-REG is partitioned into these exact classes, and furthermore that it is decidable at which level some ω-regular language resides. We include the visual representation of Wagner's hierarchy on page 13.

Another way in which Wagner's hierarchy advanced the study of finite state machine recognizable languages of Σ^ω was in its refinement of the corresponding relativized Borel hierarchy, allowing for a finer notion than that provided by descriptive set theory of how "complicated" a language is.

Zooming out, we can also consider the *downwards* complexity classes presented in [11], denoted $\widehat{G}_m^n$ for $G \in \{C, D, E\}$. Essentially, they contain languages "reducible" to the corresponding exact class. We look at these generalized classes because of their equivalence with the low level Borel classes we looked at in section 1, thus giving a familiar frame of reference in studying Wagner's classes as well as showing a compatibility between complexity in terms of set-theoretic definability, and complexity in terms of structural, automata-theoretic properties.

As mentioned, this paper in particular will look at the class $\widehat{C}_2^1$, Wagner's equivalent to the Π_2^0 or G_δ languages of ω-REG. We give a proof of the equivalence, in one direction analyzing a G_δ regular language with the chain and superchain measures. Doing so sheds light on the necessary properties (and limitations) of automata accepting exactly this class. This interpretation will then be used to help present the main proof in [1], covered in section 4.

But first, we need to provide the necessary machinery for understanding this measure.

3.1 The Chain and Superchain measures

We begin by identifying the subsets of Q which are "practical" in the sense that they only contain states which can potentially be visited by the automaton, otherwise called *accessible*, and that these states form loops or cycles and thus could be seen over and over in an infinite sequence of states (i.e., a run of the automaton). We therefore define two notions :

Definition 8. *Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F}$*[4] *be a finite state automaton. A state $q \in Q$ is called* ***accessible*** *if there exists a word $x \in \Sigma^*$ such that $\delta(q_0, x) = q$.*

Recalling some graph theory,

Definition 9. *Let $G = (V, E)$ be a directed graph. A* ***strongly connected set*** *is a subset $S \subseteq V$ such that for any $u, v \in S$ there exists a path in G from u to v and likewise from v to u.*

4. We remark that this definition is not exclusive to automata on finite or infinite words, determinism, nor the mode of acceptance

*A **strongly connected component** S is a strongly connected set which is maximal ; i.e., there is no other strongly connected set S' such that $S \subsetneq S'$*

Definition 10. *For a deterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ we define the family of **essential sets** :*

$$\mathbb{M}(\mathcal{A}) := \{F \subseteq Q \mid F \text{ is a strongly connected component of } Q\} \tag{10}$$

Intuitively, the name "essential" alludes to the property that these are sets of states which can be seen over and over again (i.e. those with loops or cycles). They are sometimes called *admissable* sets, and in section 4 we will see yet another characterization of them as sets of *realizable cycles* ([7]).

In particular we note the set $\mathrm{Inf}(\sigma)$ is an essential set for any run σ by a finite state machine. The following lemma outlines the nice properties of essential sets interpreted by language and automata-theoretic ideas as prefixes and intermediate transitions.

Lemma 1. *$S \in \mathbb{M}(\mathcal{A})$ if and only if there is $q \in S$ and finite words $x, y \in \Sigma^*$ such that*

$$\delta(q_0, x) = \delta(q, y) = q;$$
$$S = \{\delta(q, w) \mid w \sqsubseteq y\}$$

In fact, for any state q belonging to an essential set S one can find words $x, y \in \Sigma^*$ such that the above conditions hold.

In other words, the essential sets are the sets of states such that if there is a cycle $(q, q_1, ..., q_n, q)$ in $\mathcal{A}$, accessible from the start state q_0, then the set $\{q, q_1, ...q_n\}$ is essential. The essential sets are those which can give rise to a run which is either accepting or rejecting since they will influence the set $\mathrm{Inf}(\sigma)$. We divide the essential sets accordingly, based on the acceptance component $\mathcal{F}$ of a deterministic Muller automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$. [5]

$$\mathbb{M}^+(\mathcal{A}) := \mathbb{M}(\mathcal{A}) \cap \mathcal{F}$$
$$\mathbb{M}^-(\mathcal{A}) := \mathbb{M}(\mathcal{A}) \cap \mathcal{F}^C$$

From here, we can now look for *chains* of elements of $\mathbb{M}(\mathcal{A})$: increasing (with respect to $\subseteq$) sequences of essential sets that alternate between accepting ($\mathbb{M}^+(\mathcal{A})$ and rejecting ($\mathbb{M}^-(\mathcal{A})$. Chains are distinguished by the "polarity" of the innermost set :

Definition 11. *A **positive** chain (resp. **negative** chain) is a sequence $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ for some $n \in \omega$, such that $F_1 \in \mathbb{M}^+(\mathcal{A})$ (resp. $F_1 \in \mathbb{M}^-(\mathcal{A})$).*

Now we focus our attention on the maximal lengths of chains, sorting the chains of $\mathcal{A}$ via the inductively defined sets $M_m^+(\mathcal{A})$ and $M_m^-(\mathcal{A})$: for a set of states $F \in Q$, let $S \in M_m^+(\mathcal{A})$ (resp. $M_m^-(\mathcal{A})$ if and only if F is the last set of a positive chain (resp. negative chain) of length m.

$$M_1^+(\mathcal{A}) := \mathbb{M}^+(\mathcal{A}) \qquad\qquad M_1^-(\mathcal{A}) := \mathbb{M}^-(\mathcal{A})$$

We use Wagner's notation, $m^+(\mathcal{A})$ (resp. $m^-(\mathcal{A})$) to denote the largest $m < \omega$ such that the set $M_m^+(\mathcal{A})$ (resp. $M_m^-(\mathcal{A})$) is nonempty. By convention we let $m^+(\mathcal{A})$ ($m^-(\mathcal{A})$) equal 0 in the case the above two sets are empty.
We sometimes will abuse notation and write m^+ or M_m^+ instead of m^+, M_m when the automaton $\mathcal{A}$ is clear from the context.
To see things symbolically :

Definition 12. *For natural number $m \geq 1$,*

$$M_{2m}^+ := \{F \in \mathbb{M}^-(\mathcal{A}) \mid \exists F' \in M_{2m-1}^+ \text{ s.t. } F' \subseteq F\} \qquad M_{2m}^- := \{F \in \mathbb{M}^+(\mathcal{A}) \mid \exists F' \in M_{2m-1}^- \text{ s.t. } F' \subseteq F\}$$
$$M_{2m+1}^+ := \{F \in \mathbb{M}^+(\mathcal{A}) \mid \exists F' \in M_{2m}^+ \text{ s.t. } F' \subseteq F\} \qquad M_{2m+1}^- := \{F \in \mathbb{M}^-(\mathcal{A}) \mid \exists F' \in M_{2m}^- \text{ s.t. } F' \subseteq F\}$$

And we let

$$m^+(\mathcal{A}) := \max(\{m \mid M_m^+ \neq \emptyset\} \cup \{0\}) \qquad\qquad m^-(\mathcal{A}) := \max(\{m \mid M_m^- \neq \emptyset\} \cup \{0\})$$

5. We do this without loss of generality by MacNaughton's theorem : any automaton accepting $L \in \omega$-REG can be transformed into deterministic Muller. However, we will later see that the invariance property of the chain measures renders this remark inconsequential.

The inductive construction of these sets together with the fact $\mathbb{M}(A)$ is a finite set gives us a decidability result :

Lemma 2. *For any given $\mathcal{A}$, a deterministic Muller automaton on ω-words, there exist algorithms to compute $m^+(\mathcal{A})$ and $m^-(\mathcal{A})$.*

In particular we refer the reader to [14] for an efficient algorithm to compute the maximal positive chain number $m^+(\mathcal{A})$.

The sets $M^\pm_m$ have the following properties :

Lemma 3.
1. *$F \in M^+_m \quad \Leftrightarrow \quad \exists F_1, F_2, ..., F_m \subseteq Q$ such that $F_m = F$, $F_i \subseteq F_{i+1}$, and $F_i \in \mathbb{M}^+(\mathbb{M}^-)$ for $i \leq m$ odd (even).*
2. *$F \in M^-_m \quad \Leftrightarrow \quad \exists F_1, F_2, ..., F_m \subseteq Q$ such that $F_m = F$, $F_i \subseteq F_{i+1}$, and $F_i \in \mathbb{M}^+(\mathbb{M}^-)$ for $i \leq m$ even (odd).*
3. *For m even, $M^+_m \subseteq M^-$ and $M^-_m \subseteq M^+$*
4. *For m odd, $M^+_m \subseteq \mathbb{M}^+$ and $M^-_m \subseteq \mathbb{M}^-$.*
5. *$|m^+ - m^-| \leq 1$*
6. *For automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$, $m^\pm(\mathcal{A}) = m^\mp(\mathcal{A}^C)$ where $\mathcal{A}^C = (Q, \Sigma, \delta, q_0, \mathcal{F}^C)$.*

We now introduce the **superchain measure**. It is defined in a similar way as above, but instead of measuring sequences of elements $F \in \mathbb{M}$ related by set inclusion, the *superchains* are alternating sequences of maximal length chains– i.e., chains of chains. The superchains are ordered the relation of *reachability* :

Definition 13. *For a set of states Q and subsets $F_1, F_2 \subseteq Q$, a state $q_2 \in F_2$ is **reachable** from $q_1 \in F_1$ if and only if there exists a finite word $w \in \Sigma^*$ such that $\delta(q_1, w) = q_2$.*
We say the set F_2 is reachable from F_1 if $\exists q_2 \in F_2$ and $\exists q_1 \in F_1$ such thatq_1 is reachable for q_2. In particular, when F_1, F_2 are essential sets, F_2 is reachable from F_1 if any *$q_2 \in F_2$ is reachable from* any *$q_1 \in F_1$.*
This relation is denoted $F_1 \trianglerighteq_{\mathcal{A}} F_2$ (or $F_1 \trianglerighteq F_2$ when $\mathcal{A}$ is clear from the context).

We build by induction on n the sets $N^+_n(\mathcal{A})$ and $N^-_n(\mathcal{A})$, restricting our attention to chains of maximal length.
From here on, we use $\mathfrak{m}$ to denote the natural number $\mathfrak{m} = \max(m^+(\mathcal{A}), m^-(\mathcal{A}))$.

$$N^+_1(\mathcal{A}) := M^+_{\mathfrak{m}}(\mathcal{A}) \qquad\qquad N^-_1(\mathcal{A}) := M^-_{\mathfrak{m}}(\mathcal{A})$$

For $n < \omega$, we define sets of superchains :

$$N^+_{2n} := \{F \in M^-_{\mathfrak{m}}(\mathcal{A}) \mid \exists F' \in N^+_{2n-1} \text{ s.t. } F' \trianglerighteq F\} \qquad N^-_{2n} := \{F \in M^+_{\mathfrak{m}}(\mathcal{A}) \mid \exists F' \in N^-_{2n-1} \text{ s.t. } F' \trianglerighteq F\}$$
$$N^+_{2n+1} := \{F \in M^+_{\mathfrak{m}}(\mathcal{A}) \mid \exists F' \in N^+_{2n} \text{ s.t. } F' \trianglerighteq F\} \qquad N^-_{2n+1} := \{F \in M^-_{\mathfrak{m}}(\mathcal{A}) \mid \exists F' \in N^-_{2n} \text{ s.t. } F' \trianglerighteq F\}$$

We set

$$n^+(\mathcal{A}) := \max(\{m \mid N^+_n(\mathcal{A}) \neq \emptyset\} \cup \{0\}) \qquad\qquad n^-(\mathcal{A}) := \max(\{m \mid N^-_n(\mathcal{A}) \neq \emptyset\} \cup \{0\})$$

I.e., n^+ and n^- are the maximal length of positive and negative superchains, respectively.

The following lemma will be used in later proofs and captures the interplay between the maximal length chains and superchains :

Lemma 4.
1. *$m^+(\mathcal{A}) = m^-(\mathcal{A}) + 1 \quad \Leftrightarrow \quad n^+(\mathcal{A}) = 1 \quad \wedge \quad n^-(\mathcal{A}) = 0$*
2. *$m^-(\mathcal{A}) = m^+(\mathcal{A}) + 1 \quad \Leftrightarrow \quad n^+(\mathcal{A}) = 0 \quad \wedge \quad n^-(\mathcal{A}) = 1$*
3. *$m^+(\mathcal{A}) = m^-(\mathcal{A}) \quad \Leftrightarrow \quad n^+(\mathcal{A}), n^-(\mathcal{A}) \geq 1$*

3.2 Exact complexity classes

Now that we have a grasp of what the measures $m^+(\mathcal{A}), m^-(\mathcal{A}), n^+(\mathcal{A}), n^-(\mathcal{A})$ actually count, we can begin to investigate how ω-REG can be classified with respect to these numbers.

To this end, we extract the pure structural (chain) properties by considering two languages to be equivalent if they are equivalent by following ordering $\leq$ on ω regular languages $U, V \subseteq \Sigma^\omega$ where for some for some finite state automata $\mathcal{A}$, $\mathcal{B}$, $U = L(\mathcal{A})$ and $V = L(\mathcal{B})$.

$$\begin{aligned} &U \leq V \Leftrightarrow \\ \mathfrak{m}(\mathcal{A}) < \mathfrak{m}(\mathcal{B}) \quad \vee \quad &[\mathfrak{m}(\mathcal{A}) = \mathfrak{m}(\mathcal{B}) \\ &\quad \wedge (n^+(\mathcal{A}) \leq n^+(\mathcal{B})) \wedge (n^-(\mathcal{A}) \leq n^-(\mathcal{B}))] \end{aligned}$$

The order $\leq$ is clearly reflexive and transitive. We pose the natural equivalence relation $\equiv$ on the ω-regular subsets of Σ^ω by letting $U \equiv V \Leftrightarrow [U \leq V \wedge V \leq U]$; the resulting equivalence classes are partially ordered by $\leq$.

These classes of ω-REG $\setminus \equiv$ are the *exact complexity classes* we are interested in. They are given the names C^n_m, D^n_m, and E^n_m. On the next page is the visual representation of what we call the **Wagner hierarchy**. It is interpreted as such : for two exact classes P, Q, if P and Q are connected and P is not higher than Q then $P \leq Q$. For example, we have $C^2_2 \leq E^2_2 \leq D^3_2$.

$$C^n_m := \{L(\mathcal{A}) \mid \mathfrak{m}(\mathcal{A}) = m \quad \wedge \quad n^+(\mathcal{A}) = n-1 \quad \wedge \quad n^-(\mathcal{A}) = n\}$$

$$D^n_m := \{L(\mathcal{A}) \mid \mathfrak{m}(\mathcal{A}) = m \quad \wedge \quad n^+(\mathcal{A}) = n \quad \wedge \quad n^-(\mathcal{A}) = n-1\}$$

$$E^n_m := \{L(\mathcal{A}) \mid \mathfrak{m}(\mathcal{A}) = m \quad \wedge \quad n^+(\mathcal{A}) = n^-(\mathcal{A}) = n\}$$

Note that we still have not seen that the measures m, n are indeed independent of the specific automaton used, and so at this point we wonder if some ω-regular language L could reside in more than one of the above classes. For example, if there were two different automata $\mathcal{A}$ and $\mathcal{B}$ (on the same language Σ) such that $L = L(\mathcal{A}) = L(\mathcal{B})$ but$\mathfrak{m}(\mathcal{A}) \neq \mathfrak{m}(\mathcal{B})$, the equivalence class of $L(\mathcal{A})$ cannot be the same of $L(\mathcal{B})$, though they are both equal to L. We will show this cannot be the case.
We give some fairly immediate properties of the exact classes. Proofs are given in [11] and follow directly from properties of the sets $M^{\pm}_m$, $N^{\pm}_n$.

Lemma 5. *1. For any ω-regular L, $L \in \bigcup_{1 \leq n,m < \omega} (C^n_m \cup D^n_m \cup E^n_m)$*

2. C^n_m and D^n_m are dual classes for all $m, n < \omega$: $L \in C^n_m \quad \Leftrightarrow \quad L^C \in D^n_m$

3. The classes E^n_m are closed under complementation : $L \in E^n_m \quad \Leftrightarrow \quad L^C \in E^n_m$

To prove this hierarchy is proper, Wagner [11] constructs two languages V^n_m and U^n_m of ω-words over the alphabet $\Sigma = \{0, 1\}$, and belonging respectively to C^n_m and D^n_m. Once these languages are classified, then, we will present Wagner's proof that V^n_m cannot be in D^n_m. Since these are equivalence classes, if we show they are not equal, then they are disjoint and the hierarchy is proper.

First, we need to see these sets are ω-regular and indeed belong to the desired classes. To this end we provide the transition graph of a specific deterministic Muller automaton $\mathcal{B} = (Q_\mathcal{B}, \{0, 1\}, \delta_\mathcal{B}, q^1_0, \mathcal{F}_\mathcal{B})$. It is pictured below in Figure 3.
We'll now define the languages V^n_m and U^n_m.

In an effort to simplify notation, first let

$$\overrightarrow{k} := (10 \cup 110 \cup ... \cup 1^k 0)$$

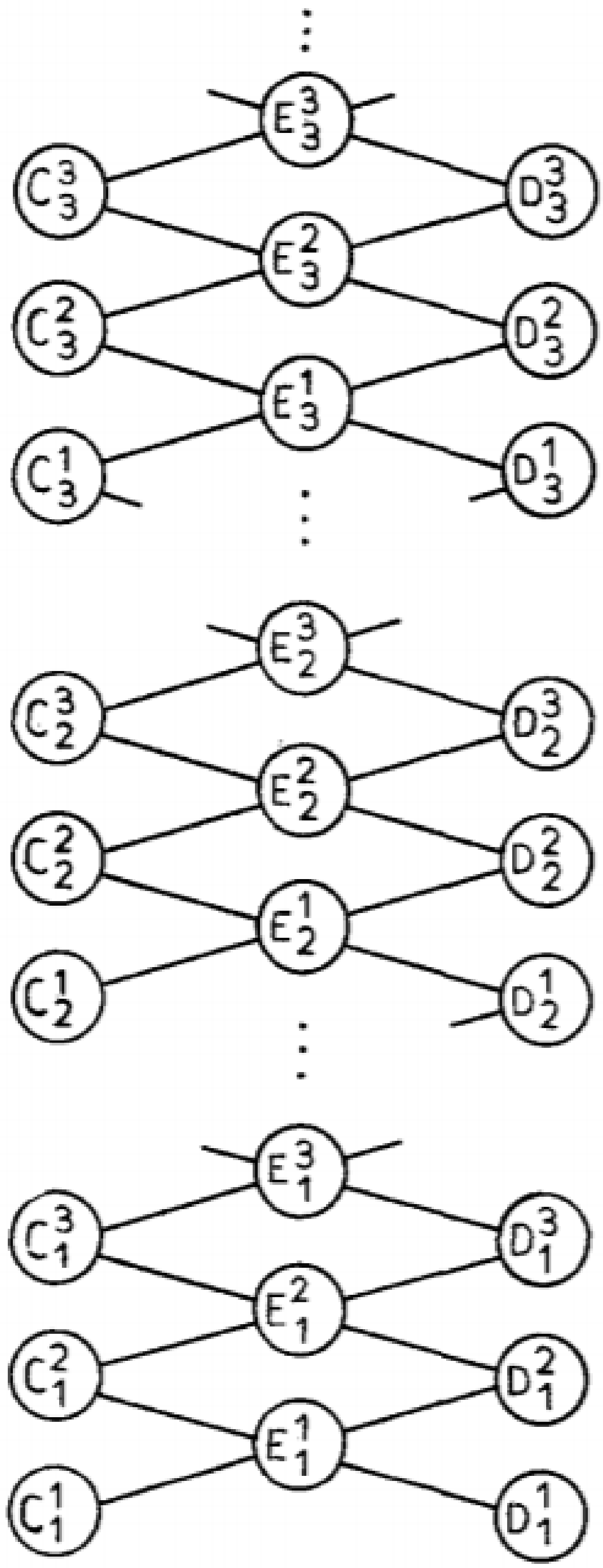
$$V^n_m := \bigcup_{\substack{0 \leq v < n \\ 1 \leq u \leq m \\ v+u even}} [\overrightarrow{m}^*.0]^v.(\overrightarrow{m}^*).[\overrightarrow{u-1}.(1^u 0)]^\omega \tag{11}$$

In this case (n odd) we let $U^n_m := (V^n_m)^C$.

For n even, we have an almost identical definition of U^n_m, yet note that the parity of $u + v$ has changed :

$$U^n_m := \bigcup_{\substack{0 \leq v < n \\ 1 \leq u \leq m \\ v+u odd}} [\overrightarrow{m}^*.0]^v.(\overrightarrow{m}^*).[\overrightarrow{u-1}.(1^u 0)]^\omega \tag{12}$$

Similarly, $V^n_m := (U^n_m)^C$, whence n even.



E_3^3
C_3^3
D_3^3
E_3^2
C_3^2
D_3^2
E_3^1
C_3^1
D_3^1
E_2^3
C_2^3
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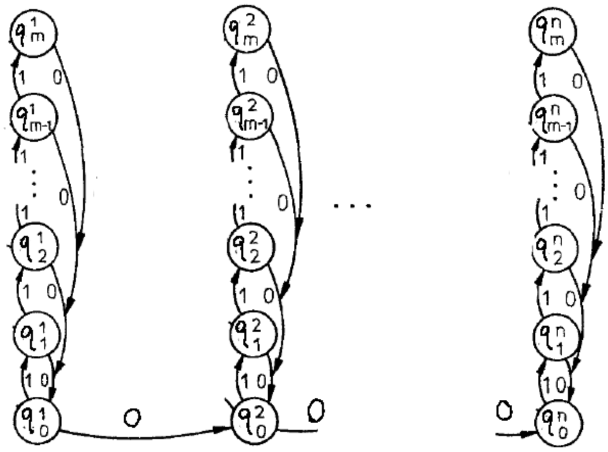


FIGURE 3 – The automaton $\mathcal{B}$ recognizing V_m^n for n odd.

Now we must define a suitable acceptance component to ensure $\mathcal{B}$ to recognize V_m^n, for n odd ; the case for n even is similar.
We set

$$\mathcal{F}_{\mathcal{B}} = \{\{q_0^v, ..., q_u^v\} \mid 0 < u \leq m \quad \wedge \quad 0 < v \leq n \quad \wedge \quad u + v \text{ odd}\}.$$

Observing some characteristics about the structure of $\mathcal{B}$ is also a way to develop a visual intuition for how the chains and superchains of a language are observable in the structure of a finite state automaton.
Example

Consider the set $F_1 = \{q_0^1, q_1^1\}$. Let σ be a run by $\mathcal{B}$ such that $\text{Inf}(\sigma) = F_1$. Using the transition graph, we see that the run σ could only arise from $\mathcal{B}$ reading a word of the form $(10)^\omega$, which is not in V_m^n. Correspondingly, F_1 is a rejecting cycle ; $F_1 \in M^-(\mathcal{B})$. On the other hand, $F_1 \subseteq \{q_0^1, q_1^1, q_3^1\} \in \mathcal{F}$, so $m^-(\mathcal{B}) \geq 2$.

The transition graph for $\mathcal{B}$ above visually organizes the chains and superchains : chains are captured by the vertical columns each of height m ; as we just saw in the previous example, going up a level corresponds to increasing the chain length. For v odd, the largest index u in essential set $F = \{q_0^v, q_1^v, ..., q_u^v\}$ indicates that $F \in M_u^+(\mathcal{B})$ for u odd, and $F \in M_u^-(\mathcal{B})$ for u even. As $1 \leq u \leq m$, $\mathfrak{m}(\mathcal{B}) = m$.

Adding 1 to n^+ or n^- corresponds to the automaton moving one column to the right. The transitions between columns labeled by 0 is observable in the definition (11) by the 0 in the factor $(\overrightarrow{m}^*.0)^v$; each time the top index v is increased by 1, the automaton must read another 0 from state $\delta(q_0^v, \overrightarrow{m}) = q_0^v$, and $\delta(q_0^v, 0) = q_0^{v+1}$ (for $1 \leq v \leq n$), hence the machine has moved to the right one column. The parity change in v gives the alternating behavior of the superchain.

Note that not all runs by $\mathcal{B}$ on words in V_m^n will witness maximal chains and superchains ; the word $(10.110)^\omega \in V_m^n = L(\mathcal{B})$ is accepted by a run which sees only a length 2 negative chain. Providing a particular case of a run of $\mathcal{B}$ that realizes maximal chains and superchains is not crucial for the proof $V_m^n \in C_m^n$, since it suffices to show there exists some finite state Muller automaton capable of computing V_m^n.

However in the proof of Theorem 5 to follow, that $V_m^n \notin D_m^n$, we need to show that for an arbitrary automaton $\mathcal{A}$ recognizing V_m^n, $L(\mathcal{A}) \notin D_m^n$. As was mentioned after the definition of the exact complexity classes, we haven't yet shown that any automaton equivalent to our particular $\mathcal{B}$ will have the same number of maximal chains and superchains. Thus not only do we need to generalize the automata we work with in the proof, but we will need a careful construction of an ω-word complex enough that a maximal length chains and superchains will be realized by an automaton with sufficient expressive power. Specifically, since the difference between classes C_m^n and D_m^n is the number of superchains, we will need this "witness" word to force the any appropriately powerful machine into running through a negative superchain length n.

Formalizing this discussion, from $\mathcal{B}$ of 3.2 we conclude :

Lemma 6. *1.* $V_m^n \in C_m^n$

2. $U_m^n \in D_m^n$

Proof. It was explained above how 3.2 suffices as a proof. The construction of this specific automaton is credited to Wagner [11]. □

Theorem 5. *$V_m^n \notin D_m^n$ for n odd*

Proof. We treat the case where m is also odd, as Wagner does, but we focus on filling in the details missing in the original proof. Since V_m^n is indeed an ω-regular language, allowing us then to apply McNaughton's theorem to take an arbitrary deterministic Muller automaton $\mathcal{A} = (Q, \{0,1\}, \delta, q_0, \mathcal{F})$ such that $L(\mathcal{A}) = V_m^n$.

Denote $s = |Q| < \omega$.

We construct by recursion on m a finite word w_m, our so called "witness" word.

$$\begin{aligned} w_1 &:= (1^1.0)^s \\ w_2 &:= (1^2.0.w_1)^s \\ &\dots \\ w_m &:= (1^m.0.w_{m-1})^s \end{aligned}$$

Now we consider runs by $\mathcal{A}$ on the infinite word $(w_m)^\omega$. By the way the prefixes $(w_m)^\omega$ have been defined, the acceptance or rejection of a run will depend on the parity of m, so a run on $(w_m^\omega) = (1^m 0.w_{m-1})^\omega = \dots = 1^m 0.(1^{m-1}0.(\dots(1^2 0(10)^s)^s)..)^s)^\omega$, $\mathcal{A}$ must realize a chain of length m.

We'll need the following :

Claim 1. *$(w_m)^\omega \notin V_m^n$ for m odd.*

Proof. We try to rule out possible candidates of words in V_m^n to show that $(w_m)^\omega$ can not possibly be one of them. Let's call a word "good" if it is a member of V_m^n and of the form $(1^m.0.w_{m-1})^\omega$.

The infinite word $(w_m)^\omega = (1^m.0.w_{m-1})^\omega$ has an infinite number of factors $1^m.0$, so we must take $u = m$ in the definition of the language V_m^n. Accordingly, since m odd, the parameter $v \in [0, n)$ must also be odd, and any "good" word will be in the sublanguage :

$$\bigcup_{\substack{0<v<n \\ v odd}} [\overrightarrow{m}^*.0]^v.(\overrightarrow{m}^*).[\overrightarrow{m-1}.(1^m)]^\omega$$

Since $v \neq 0$, a "good" word will always have some prefix of the form $(\overrightarrow{m}^*.0)$. Then, no matter how many (finite) iterations of $\overrightarrow{m}$, a "good" word either has consecutive 0's in its prefix, or it begins with a 0. But by the definition of w_m, this can never be the case in the word $(w_m)^\omega$.

Lastly, we remark that for m even, the word $(w_m)^\omega \in (\overrightarrow{m})^*.(\overrightarrow{m-1}.1^m.0)^\omega$, i.e. the language V_m^n when taking $v = 0$ and $u = m$. □

Since $V_m^n = L(\mathcal{A})$, by the previous claim, we know a run σ by $\mathcal{A}$ on $(w_m)^\omega = (1^m 0.w_{m-1})^\omega$ will be rejecting. Thus there exists some essential set $F_m^1 \in \mathcal{F}^C$ with $\mathrm{In}(\sigma) = F_m^1$.

We wish to show that $F_m^1 \in M_m^-(\mathcal{A})$. To do this, by Lemma 3, it suffices to find essential sets $F_1^1, F_2^1, \dots, F_{m-1}^1$ subsets of F_m^1 such that $F_i \subseteq F_{i+1}$, and $F_i \in M^+(\mathcal{A})[M^-(\mathcal{A})]$ for i even [odd].

To get a better handle on elements of F_m^1 we set $q_1 := \delta(q_0, w_m) = \delta(q_0, (1^m 0.w_{m-1})^s)$. Because of the choice of $s \geq |Q|$, at some point when reading the word w_m the automaton $\mathcal{A}$ will pass the same state q_1. Let $s_m < s$ denote this point.

Therefore,

$$q_1 = \delta(q_0, (1^m 0.w_{m-1})^{s_m} = \delta(q_1, (1^m 0.w_{m-1})^{s-s_m})$$

In other words, the states seen reading $(1^m 0.w_{m-1})^{s-s_m}$ form a cycle around the state q_1. By the choice of the essential set F_m^1, and by Lemma 1, we can see that

$$F_m^1 = \{\delta(q_1, w) \mid w \sqsubseteq (1^m 0.w_{m-1})^{s-s_m}\}$$

Now we look for an essential accepting set (which will be denoted F_{m-1}^1 contained in F_m^1. Such a set would be be composed of the transitions given by δ when $\mathcal{A}$ reads finite prefixes of a word in V_m^n (an accepting word), and this word in turn necessarily a prefix of w_m^ω so to ensure $F_{m-1}^1 \subseteq F_m^1$.

Using the recursive construction of w_m we note that we can rewrite w_m as

$$\begin{aligned} w_m &= (1^m 0.w_{m-1})^s \\ &= (1^m 0.w_{m-1})^{s-1}.(1^m 0).(w_{m-1}) \\ &= (1^m 0.w_{m-1})^{s-1}.(1^m 0).(1^{m-1}0.w_{m-2})^s \end{aligned}$$

Letting $t_m = (1^m 0 w_{m-1})^{s-1}.(1^m 0)$, and using the above claim, we find $t_m.(w_{m-1})^\omega \in V_m^n$. Therefore, when $\mathcal{A}$ reads $t_m.(w_{m-1}^\omega$ from start state q_0, there is indeed an essential set $F_{m-1}^1 \in M^+(\mathcal{A})$ consisting of states resulting from transitions from q_1 while reading prefixes of w_{m-1}^ω. These states are found more explicitly by using the previous argument for w_m^ω : the fact that $|w_{m-1}| \geq s \geq |Q|$, and by the choice of $q_1 = \delta(q_0, w_m)$, there is a moment $s_{m-1} < s$ such that

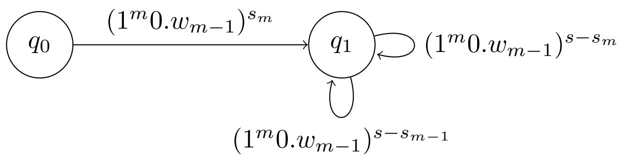
$$\begin{aligned} q_1 = \delta(q_0, w_m) &= \delta(q_0, t_m.(w_{m-1})) \\ &= \delta(q_0, t_m.(1^{m-1}0.w_{m-2})^s) \\ &= \delta(q_0, t_m.(1^{m-1}0.w_{m-2}^{s_{m-1}}) \\ &= \delta(q_1, t_m.(1^{m-1}0.w_{m-2})^{s-s_{m-1}}) \end{aligned}$$

Now we can apply similar reasoning to F_{m-1}^1 to see :

$$F_{m-1}^1 = \{\delta(q_1, w) \mid w \sqsubseteq (1^{m-1}0.w_{m-2})^{s-s_{m-1}}))\}$$

From these first two iterations, we start to get a sense of the nested cycles originating at the state q_1. By definition, $F_{m-1}^1 \subseteq F_m^1$.

The decomposition $w_m = t_m.(1^{m-1}.w_{m-2})^s = t_m.t_{m-1}(1^{m-2}0.w_{m-3})^s$,
where we let $t_{m-1} := (1^{m-1}0.w_{m-2})^{s-1}.(1^{m-1})$ gives the next step of the proof.



Iterating this argument until $m = 1$, we obtain

$$F_1^1 \subseteq F_2^1 \subseteq ... \subseteq F_{m-1}^1 \subseteq F_m^1 \tag{13}$$

Furthermore, $F_i^1 \in \mathbb{M}^-(\mathcal{A})$ for i odd ; in particular we have $F_1^1 \in \mathcal{F}^C$ and conclude that (13) is a negative chain of $\mathcal{A}$ of length m, and so $m^-(\mathcal{A}) \geq m \Rightarrow \mathfrak{m} \geq m$.
If $\mathfrak{m} > m$, then $L(\mathcal{A}) = V_m^n \notin D_m^n$ and we are done.

So suppose not, $\mathfrak{m} = m$. Our goal is to now construct a negative superchain : using from the negative chain (13) as our first element, we look for alternating positive/negative chains each of maximal length $\mathfrak{m} = m$. The top index j of F_i^j is used to keep track of where we are along the superchain. Once we have found a collection of n such chains $(F_i^j)_{1 \leq i \leq m}$, we then need to check that for all $1 \leq j \leq n$, F_i^{j+1} is reachable from F_i^j. Assuming all this can be done, the existence of a negative superchain of length n then gives the result.
Construction of the superchain : We now consider runs by $\mathcal{A}$ on the word $w_m.0.(w_m)^\omega$.
Letting $q_2 := \delta(q_0, w_m.0.w_m)$, Since $(w_m).0.(w_m)^\omega = (1^m 0.w_{m-1}.0).(1^m 0 w_{m-1})^\omega$ is not a word of V_m^n by the claim, a run on $(w_m.0.w_m)^\omega$ is rejecting. Thus we find an essential set F_m^2 and a finite word x such that

$$F_m^2 = \{\delta(q_0, w) \mid w \sqsubseteq x\}$$

After $n-2$ more iterations of this argument we get the family :

$$\{F_j^i \mid F_j^i \subseteq F_{j+1}^i \quad 1 \leq j < m\}_{1 \leq i \leq n} \tag{14}$$

where for i odd, $(F_j^i)_{1 \leq j < m}$ is a negative chain, and is a positive chain for i even. We conclude that there is a negative superchain of length n in $L(\mathcal{A}) = V_m^n$, as desired. □

As the automaton $\mathcal{A}$ was arbitrary up to recognizing the language V_m^n, the above proof immediately gives the following corollary stating the invariance property of the chain and superchain measures.

Corollary 2. *For any two automata $\mathcal{A}$, $\mathcal{B}$ recognizing a language L, $\mathfrak{m}(\mathcal{A}) = \mathfrak{m}(\mathcal{B})$, $n^+(\mathcal{A}) = n^-(\mathcal{B})$, and $n^-(\mathcal{A}) = n^-(\mathcal{B})$.*

3.3 Downwards Complexity Classes and their Topological Interpretations

Coarser than the exact complexity classes are the *"downward"* complexity classes, still defined by the same measures. The following definitions of, and subsequent lemmas pertaining to, these classes put into perspective the precision of the exact Wagner hierarchy in comparison with the topological correspondences we have seen.

$$\widehat{C^n_m} := \{L(\mathcal{A}) \mid \max((m^+(\mathcal{A}), m^-(\mathcal{A})) < m) \quad \vee ((m^+(\mathcal{A}), m^-(\mathcal{A})) = m) \quad \wedge n^+(\mathcal{A}) \leq n-1)\}$$

$$\widehat{D^n_m} := \{L(\mathcal{A}) \mid \max((m^+(\mathcal{A}), m^-(\mathcal{A})) < m) \quad \vee ((m^+(\mathcal{A}), m^-(\mathcal{A})) = m) \quad \wedge \quad n^-(\mathcal{A}) \leq n-1)\}$$

$$\widehat{E^n_m} := \{L(\mathcal{A}) \mid \max((m^+(\mathcal{A}), m^-(\mathcal{A})) < m) \quad \vee ((m^+(\mathcal{A}), m^-(\mathcal{A})) = m) \quad \wedge n^+(\mathcal{A}) \leq n \quad \wedge n^-(\mathcal{A}) \leq n)\}$$

Recalling the order $\leq$ we can further characterize these classes :

Lemma 7. *1.* $L \in \widehat{C^n_m} \Leftrightarrow L^C \in \widehat{D^n_m}$

2. $L \in \widehat{G^n_m} \Leftrightarrow \exists U(U \in G^n_m \wedge L \leq U)$ *for* $G \in \{D, C, E\}$

Roughly, we can view the downwards classes as those ω-regular languages which are less complex than a language belonging to the corresponding exact class. The above classes have topological analogues ; the proofs have been done by Wagner in [13].

Theorem 6. *1.* $\Sigma^{0\mathcal{R}}_1 = \widehat{C^2_1}$

2. $\Pi^{0\mathcal{R}}_1 = \widehat{D^2_1}$

3. $\Pi^{0\mathcal{R}}_2 = \widehat{C^1_2}$

4. $\Sigma^{0\mathcal{R}}_2 = \widehat{D^1_2}$

5. $\Delta^{0\mathcal{R}}_2 = \bigcup_{n=1}^{\infty}(\widehat{C^n_1} \cup \widehat{D^n_1})$

We give the proof of (3) because of the role Π^0_2 languages play in [1].
The next lemma gathers a handful of equivalent representations of a G_δ ω-regular language, and so will be used in proving Theorem 5.3. or we can make use of the following notions :

Lemma 8. *The following are logically equivalent for ω-Regular languages L :*

1. $L \in \Pi^0_2$ *;*

2. L is recognized by a deterministic Büchi automaton ;

3. L is recognized by a full table Muller automaton ;

4. $L = \overrightarrow{X}$ *for some* $X \subseteq \Sigma^\omega, X \in REG$.

Proof (Theorem 6.3). Supposing $L \in \widehat{C^1_2}$, we must consider the possible values of $\mathfrak{m}$ and $n^\pm$ for a deterministic Muller automaton $\mathcal{A}$ such that $L(\mathcal{A}) = L$.

Case I :

Suppose $\mathfrak{m} < 2$. The conclusion is trivial for $\mathfrak{m} = 0$, so let $\mathfrak{m} = 1$. Then $\mathcal{A}$ has any chain, positive or negative, being at most length 1. This implies the sets $M^+(\mathcal{A})$ and $M^-(\mathcal{A})$ are incomparable in the sense that $\forall F, F' \in M^+, M^-$ respectively, we never have $F \subseteq F'$ or $F' \subseteq F$. In particular, no accepting is contained in a rejecting set, hence $\mathcal{A}$ is full-table, so $L(\mathcal{A}) \in \Pi^0_2$.

Case II :

Suppose now $\mathfrak{m} = 2$, requiring $n^+ \leq 0$. This means the deterministic Muller automaton $\mathcal{A}$ has either a positive or a negative chain (or both) of length 2, yet has no positive superchain.

We consider further subcases :

Case II.i

If $\mathfrak{m} = m^+ = m^- = 2$, Lemma 4.3 implies $n^+, n^- \geq 1$, but that is incompatible with $L \in \widehat{C^1_2}$ by definition.

Case II.ii

Suppose it is a positive chain of $\mathcal{A}$ which realizes the maximal chain length, and that $m^- = 1$. Note that by Lemma 3.5 m^- cannot be 0. Using Lemma 4.2, $n^+ = 1$ and $n^- = 0$, so again we do not need to consider this case.

Case II.iii

By elimination we necessarily have that if $\mathfrak{m} = 2$ then $m^+ = 1$, $m^- = 2$, and Lemma 4.1 gives that $n^+ = 0$ and $n^- = 1$.

The values of m^+, m^-, and n^+ in this case imply that $\mathcal{A}$ is full-table Muller. In fact, we arrive at this conclusion just by requiring the maximal length of a positive chain be 1, as this implies that no accepting set is subsumed by rejecting one. Thus $L(\mathcal{A}) \in \Pi^0_2$. Note that when $\mathfrak{m} > m^+$, as in this case, n^+ is necessarily equal to

0 by Lemma 4.2 so we are still within the definition of $\widehat{C_2^1}$. Another small observation is that that G_δ ω-regular languages can still have negative 2-chains ; an easy example is the language $K := \{\alpha \in \{a,b\}^\omega \mid \alpha$ contains an infinite number of occurrences of the letter $a\}$.

Furthermore, no positive superchains can exist since n^+, n^- measure only superchains of chains of maximal length $\mathfrak{m}$; having no positive chain of maximal length we could not hope to find a positive superchain.
The converse is simpler : given $L \in {\Pi_2^0}^{\mathcal{R}}$, Lemma 9 gives that $L = L(\mathcal{A})$ for deterministic Büchi or full-table Muller automata. We take the latter characterization to see that $m^+(\mathcal{A})$ is necessarily 1 or 0, implying $m^-(\mathcal{A}) \leq 2$, thus $L(\mathcal{A}) \in \widehat{C_2^1}$. □

3.4 Rabin Index

In this section we will briefly introduce another measure of complexity on the ω-regular languages : the Rabin index. However, unlike the chain measure, this measure is specific to Rabin automata defined in the introduction. One can begin to draw connections between this index and the notion of a difference hierarchy– this will be made explicit in section 4. We define the **Rabin index** of an ω-regular set as the minimal number of sets (E_i, F_i) needed in the acceptance component Ω of a Rabin automaton recognizing the language (such an automaton always exists due to Rabin automata having equivalent expressive power as Muller automata). For an example of how this measure is taken, consider the transformation from a deterministic Büchi automaton $\mathcal{A}_B = (Q, \Sigma, \delta, q_0, F)$ into a determinstic Rabin $\mathcal{A}_R$. Recalling the Büchi condition (2), it is easy to see that simply setting $\mathcal{A}_R = (Q, \Sigma, \delta, q_0, \{(\emptyset, F)\})$ defines an equivalent Rabin automaton accepting $L(\mathcal{A}_B)$. We thus give this language the Rabin index 1.

Denote by $\mathrm{IR}(L)$ the Rabin Index of $L \in \omega$-REG.
Symbolically,

Definition 14. *IR(L)* $:= \min\{m \mid \exists \mathcal{A}_R = (Q, \Sigma, \delta, q_0, \Omega)$ *s.t.* $|\Omega| = m \wedge L(\mathcal{A}_R) = L\}$

$$\textit{Where } \Omega \subseteq \mathcal{P}(Q) \times \mathcal{P}(Q) \textit{ and}$$
$$L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \exists \sigma \textit{ a run on } \alpha \textit{ s.t. } \mathrm{In}(\sigma) \cap E_i = \emptyset \wedge \mathrm{In}(\sigma) \cap F_i \neq \emptyset \quad \textit{for some } 1 \leq i \leq m\}$$

Recalling that the downwards equivalence classes have very nice topological interpretations, Wagner has employed topological methods to prove the following result [11].

Wagner proves the following correspondance between maximal positive chains (m^+) and the Rabin Index.

Theorem 7. *(IR/ m^+ correspondence)*

$$IR(L(\mathcal{A})) = \lfloor (\frac{m^+(\mathcal{A}) + 1}{2}) \rfloor$$ [6]

In fact, [14] shows that given $\mathcal{A}_M$ a deterministic Muller automaton, one can compute in polynomial time $\mathrm{IR}(L(\mathcal{A}_R))$.

3.5 Decidability

We briefly include relevant decidability and computability results.

Theorem 8.
- *Given an ω-regular language L and a finite state automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$, there is an algorithm for determining at which level L resides in the Borel hierarchy*[7]
- *Given a finite state automaton $\mathcal{A}$ on ω-words over Σ, IR(L($\mathcal{A}$)) is computable.*

4 Difference Hierarchy of Δ_3^0

Let us consider again the Borel subsets of a separable, completely metrizable space, and in particular the classes Δ_α^0, also called the "ambiguous classes" .

We have just seen how Wagner's hierarchy is a refinement of this fragment of the Borel sets of $\mathcal{P}(\Sigma^\omega) \cap \omega$-REG ; a result by Hausdorff and Kuratowski gives another refinement, but of the larger more general class Δ_3^0 : the *difference hierarchy.* Roughly speaking, it gives an order on the Δ_3^0 subsets corresponding to how many G_δ (or Π_2^0) subsets of which the given Δ_3^0 set is formed. See [6] for more.

First, let us define the levels constituting the difference hierarchy :

6. $\lfloor k \rfloor$ denotes largest natural number less than k.
7. relativized to the ω-regular sets of $\mathcal{P}(\Sigma^\omega)$

Definition 15. *For a pointclass Ω of subsets of Σ^ω, let the class $D_n(\Omega)$ be defined as follows :*

$$\begin{aligned} L \in D_n(\Omega) \quad &\Leftrightarrow \quad L = (L_1 - L_2) + (L_3 - L_4) + ... \pm L_n \\ &\quad\quad \textit{where } L_i \in \Omega \quad \forall i \leq n, \\ &\quad\quad \textit{and } L_1 \supseteq L_2 \supseteq ... \supseteq L_n \end{aligned}$$

Theorem 9. *[6] Let X be a completely metrizable, separable topological space. Then for any countable non null ordinal α, we have :*

$$\Delta^0_{\alpha+1} = \bigcup_{1 \leq \alpha < \omega_1} D_n(\Pi^0_2) \tag{15}$$

As there are continuum many Borel sets belonging to the class Δ^0_3, and only countably many ω-regular languages, we wonder what happens when we relativize (15) to just those Δ^0_3 subsets computable by finite state automata (the class ω-REG).

Similar to Theorem 9, Barua [1] has defined a difference hierarchy on the ω-regular languages in terms of ω-regular G_δ languages, also characterized as being those languages computable by deterministic Büchi automata by Lemma 9. Let us call this class the "Büchi G_δ's".
Barua constructs the automata theoretic analogue of (15) in the following way : we define the class $\mathbf{D}_n$ as in Definition 12, yet we take Ω to be the class of regular G_δ languages over Σ. The first portion of [1] is dedicated to proving the following :

Since any ω-regular language L is a Δ^0_3 subset of Σ^ω, we can conclude that L resides in some level D_n in (15), and that L is a union of differences of n Π^0_2 or G_δ sets.
What is not immediate, however, was if this n coincided with the hierarchy $\bigcup_{n<\omega} \mathbb{D}_n$.
Barua was first to gave a positive answer to this in [1] for the case $n \geq 2$, and we present this proof in the following section, establishing that indeed :

$$D^{\mathcal{R}}_n = \mathbb{D}_n \tag{16}$$

4.1 Difference Hierarchy of Büchi $\mathbf{G}_\delta$s

As in the beginning of this section, we define the levels $\mathbb{D}_n$ of a difference hierarchy on ω-REG :

$$L \in \mathcal{D}_n \quad \Leftrightarrow \quad L = \bigcup (G_i - G_{i+1}) \tag{17}$$

The following is due to Barua [1]. It allows us to completely characterize ω-REG with the levels defined above.

Theorem 10. *The following are equivalent for $L \subseteq \Sigma^\omega$:*

1. *L is ω-regular*
2. *L is a finite union of differences of Büchi G_δ languages*
3. *L is a finite disjoint union of differences of Büchi G_δ languages*
4. *There exists a decreasing sequence $(G_i)_{0 \leq i \leq n}$ with $G_i \supseteq G_{i+1}$ and G_i is Büchi G_δ for all $0 \leq i \leq n$ such that*

$$L = \bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (G_i - G_{i+1}) \tag{18}$$

We briefly sketch a proof of the implication (1) $\Rightarrow$ (4) is done by showing the class $\mathbb{B}$ of languages admitting a representation (18) is a Boolean algebra containing the the Büchi G_δ sets. Given that these are exactly the class ${\Pi^0_2}^{\mathcal{R}}$, Corollary 1 implies ω-REG $\subseteq \mathbb{B}$. It is not complicated to show $\mathbb{B}$ is closed under complement and finite union. The details can be found in [1].

The above theorem allows us to conclude

Corollary 3. *ω-REG $= \bigcup_{n<\omega} \mathbb{D}_n$*

Recalling notation, let $D^{\mathcal{R}}_n$ denote the class $D_n(\Pi^0_2)$ restricted to only the ω-regular sets. As we just saw,

$$D^{\mathcal{R}}_n = D_n \cap \bigcup_{n<\omega} \mathbb{D}_n$$

What is now left to be investigated is if for each $n < \omega$, the levels D^R_n and $\mathcal{D}_n$ are the same. In other words, for a given $L \in \omega$-REG, can the length of the sequence of G_δ sets–of which L is the union of differences–the same, whether or not we restrict our attention to the ω-regular G_δ ?

Theorem 11. *Let $\mathbb{D}_n$ denote the difference hierarchy of ω-REG given in terms of Buchi G_δ languages, and let $D_n^{\mathcal{R}}$ denote the Hausdorff-Kuratowski difference hierarchy of Δ_3^0 ω-Regular sets. Then for each n,*

$$D_n^{\mathcal{R}} = \mathbb{D}_n \tag{19}$$

Clearly, $\mathbb{D}_n \subseteq D_n^{\mathcal{R}}$, therefore we will prove

$$L \in D_n^{\mathcal{R}} \Rightarrow L \in \mathbb{D}_n \tag{20}$$

The following lemma is quite useful. Note that it is not immediate because the acceptance components are families of subsets from the *same* state set Q.

Lemma 9. *Let L_1 and L_2 be two languages in ω-REG over the same alphabet Σ, and suppose $L_1 \subseteq L_2$. Then there two acceptance components $\mathcal{F}_\infty, \mathcal{F}_\in$ and a finite state automaton $\mathcal{A}_i = (Q, \Sigma, \delta, q_0, \mathcal{F}_\rangle)$ where $F_i \subseteq \mathcal{P}(Q)$ for $i \in \{1, 2\}$ such that $L_1 = L(\mathcal{A}_1)$, $L_2 = L(\mathcal{A}_2)$, and $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

Proof. The goal is to construct a *product* automaton from the deterministic Muller automata $\mathcal{B}_1$ and $\mathcal{B}_2$ accepting ω-regular languages L_1 and L_2 respectively.
Let $\mathcal{B}_1 = (Q_1, \Sigma, \delta_1, q_1, \mathcal{G}_1)$ and $\mathcal{B}_2 = (Q_2, \Sigma, \delta_2, q_1, \mathcal{G}_2)$.
We define $\mathcal{A}_i = (Q, \Sigma, \delta, q_0, \mathcal{F}_i$ as such : [8]

— $Q := Q_1 \times Q_2$
— $q_0 := (q_1, q_2)$
— $\delta((p, q), a) := (\delta_1(p, a), \delta_2(q, a)$ where $p \in Q_1$, $q \in Q_2$, and $a \in \Sigma$
— $\mathcal{F}_1 := \{F \subseteq Q_1 \times Q_2 \mid \pi_1(F) \in \mathcal{G}_1 \quad \wedge \quad \pi_2(F) \in \mathcal{G}_2\}$
— $\mathcal{F}_2 := \{F \subseteq Q_1 \times Q_2 \mid \pi_2(F) \quad \in \mathcal{G}_2$

Effectively, we have that $\mathcal{A}_i$ recognizes L_i for $i \in \{1, 2\}$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. □

4.2 Cyclic closures

Instead of using Wagner's same notion of essential sets (strongly connected components) to investigate ω-REG, Barua generalizes techniques applied by Landweber in [7], in which Landweber proved the result of Theorem 12 for $n = 1$ (Theorem 4.2 of [7]). The specific statement of the result will be given by Theorem 13. In fact, this particular case is equivalent to the characterization of ω-regular Π_2^0 languages given in Lemma 6 : $\mathbb{D}_1 = D_1^{\mathcal{R}}$ implies that the class of Büchi G_δ's ($\mathbb{D}_1$) is exactly the class of ω-regular Π_2^0 subsets of Σ^ω ($D_1^{\mathcal{R}}$).

The proof of the correspondance for $n \geq 2$ relies upon an inductive construction of acceptance components $\mathcal{F}_i$ such that equipping a finite state Muller automaton with $\mathcal{F}_i$ produces a Büchi G_δ language. From there, we can improve a separation-like theorem that will conclude Theorem 11.

Theorem 12. *[7] An language $L \subseteq \Sigma^\omega$ is G_δ if and only if there exists a language $X \subseteq \Sigma^*$ such that $\alpha \in L$ if and only if there exist finite words $x_1 \sqsubseteq x_2 \sqsubseteq ...x_i \sqsubseteq ...$ such that $x_i \in X$ and $x[i] \sqsubseteq \alpha$ for all $i < \omega$.*

Primarily, Barua defines the operation of taking the *cyclic closure* of an acceptance component $\mathcal{F} \subseteq \mathcal{P}(Q)$. In this subsection we define this operation, give some of its properties (namely, an analogue of Lemma 6.3), and see how it operates in the proof of Theorem 12. In particular, cyclic closures will influence the algorithms promised by the decidability results of Section 4.3. We now look at a theorem necessary for proving the level-correspondance between D_n^R and $\mathcal{D}_n$. It will be used to prove a sort of automata-theoretic separation theorem that will complete Theorem 11. A small remark :

We are interested in presenting the proof of this theorem, due to [1], since it extends the characterization(s) given in Lemma 9. Then recalling that ω-REG is the Boolean closure of the Büchi G_δ languages, the proof to follow provides an instance of how an automata-theoretic property of a certain class of subsets of Σ^ω can be used to establish a sort of regularity property of the class's Boolean closure. The class in question (Π_2^0) having a topological correspondance, we could wonder if similar techniques can be applied with respect to a different Polish topology such as those in [3].

As in the beginning of section 3.1, to investigate the language accepted by a finite state machine we first identify which subsets $F \subseteq Q$ are "useful", in the sense that they could be visited infinitely often during a run. Specifically, we look for *realizable cycles.*

In the following definitions we fix deterministic Muller automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$.

8. the automaton $\mathcal{A}_i$ is technically considered one machine ; the index i is to keep track of which acceptance component with which we equip it

Definition 16. *Let $q \in Q$ and $w \in \Sigma^*$ denote by $Q[q, w]$ the set of* ***intermediate transitions*** *:*

$$Q[q, w] := \{\delta(q, w[i]) \mid 0 \leq i \leq |w|\} \tag{21}$$

Fixing our attention on loops :

Definition 17. *For $q \in Q$, the set of* ***realizable cycles*** *(noted $\mathcal{C}_q$) is the set of $Q[q, w]$ such that there is a cycle around q labeled by the word w. Symbolically,*

$$\mathcal{C}_q := \{Q(q, w) \mid \delta(q, w) = q\} \tag{22}$$

Note that $\mathcal{C}_q$ is an essential set. Visually we can think of the sets $\mathcal{C}_q$ as in the figure below.
Let us now see the operation $\mathcal{F} \mapsto \widehat{\mathcal{F}}$, called the **cyclic closure** operation.

Definition 18. *For $\mathcal{F} \subseteq \mathcal{P}(Q)$, the* ***cyclic closure*** *$\widehat{\mathcal{F}}$ is the set*

$$\widehat{\mathcal{F}} := \{F_1 \cup F_2 \mid \exists q \in Q \quad \text{s.t.} F_1 \in \mathcal{F} \cap \mathcal{C}_q \quad \wedge \quad F_2 \in \mathcal{C}_q\}$$

The intuition behind the cyclic closure is that it collects those essential sets ($F \in \mathbb{M}(\mathcal{A})$) that contain some accepting set $F' \in \mathbb{M}^+(\mathcal{A})$. It is a sort of way to do a "one step extension" to a language different than $L(\mathcal{A})$, yet still keeping track of $L(\mathcal{A})$. If no such proper extension exists, we can define a new class of Muller automata, and we will show that the properties of this class are tightly related to others we have already seen.

Definition 19. *A Muller automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ is* ***cycle-closed*** *if the cyclic closure $\widehat{\mathcal{F}}$ is contained in $\mathcal{F}$.*

Indeed, the above is equivalent to the characterizations from Lemma 9 : suppose there is some $F \in \widehat{\mathcal{F}}$ such that $F \in \mathcal{F}^C$; by definition of $\widehat{\mathcal{F}}$, $F = F_1 \cup F_2$ where for some $q \in Q$, $F_1 \in \mathcal{A}_q \cap \mathcal{F}$. Then F_1 is an accepting set contained in a rejecting set, and $\mathcal{A}$ is not full-table. Conversely, if $\widehat{\mathcal{F}} \subseteq \mathcal{F}$, the remark after Definition 18 quickly implies all supersets of accepting sets are necessarily accepting.
We can then relativize Lemma 9 to our new definitions :

Theorem 13. *Let $L \subseteq \Sigma^\omega$ be an ω-regular language ; $L = L(\mathcal{A})$ for some finite state deterministic Muller $\mathcal{A}$. If $\mathcal{A}$ is cycle closed, then $L(\mathcal{A})$ is Büchi G_δ.*

Here is a rough proof : consider some finite state machine which is not cycle closed, say $\mathcal{B} = (Q, \Sigma, \delta, q_0, \mathcal{F}$ with $\mathcal{F} \subsetneq \widehat{\mathcal{F}}$. If we apply Wagner's chain measure, we can identify $F \in \widehat{\mathcal{F}} \setminus \mathcal{F}$ as a set in M_2^+. Therefore, $m^+(\mathcal{B}) \geq 2$. In the proof of Theorem 5 we saw that any language in the downwards class $\widehat{C_2^1}$ cannot have $m^+(\mathcal{B}) = 2$ so we can conclude $\widehat{\mathcal{F}} \setminus \mathcal{F} \neq \emptyset \Rightarrow L(\mathcal{B}) \notin \Pi_2^0$. However, Theorem 13 says that if we replace acceptance component $\mathcal{F}$ of We now look at the covering-like theorem and its proof, where we will see the inductive definition of a family $\mathcal{F}_n$ of acceptance components constructed in such a way to assure the "covering" we want can be done by Büchi G_δs.
To prove the main theorem of the next section we need a technical lemma about cyclic closures ; the proof is found in [1].

Lemma 10. *The operation $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ preserves set inclusion and is idempotent. I.e., for $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(Q)$,*

1. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \widehat{\mathcal{F}} \subseteq \widehat{\mathcal{G}}$
2. $\widehat{\widehat{\mathcal{F}}} = \widehat{\mathcal{F}}$

4.3 The Main Results

Theorem 14. *Let $K, L \subset \Sigma^\omega$ be two ω-regular languages over a finite alphabet Σ. Suppose there is a decreasing sequence $(G_i)_{0 \leq i \leq n}$ of subsets of $\Pi_2^0(\Sigma^\omega)$, satisfying the following :*

1. $K \subseteq G_0$
2. $G_i \cap L \subseteq G_{i+1}$ *for* $i < n$, i *even*
3. $G_i \cap K \subseteq G_{i+1}$ *for* $i < n$, i *odd*
4. $G_n \cap L = \emptyset$ *if* n *even, and* $G_n \cap K = \emptyset$ *if* n *odd*

Then there is a decreasing sequence $(H_i)_{0 \leq i \leq n}$ of Büchi G_δ sets also satisfying (1) - (4).

Démonstration. We will consider the case n is even ; when n odd the proof is similar. Given $n+1$ nested decreasing G_δ subsets satisfying (1)-(4), we show that there is an sequence of equal length with the same properties, yet all sets can actually be taken to be Büchi G_δ.
Applying Lemma 7 with $L_1 = L$ and $L_2 = K^C$, we get two acceptance components $\mathcal{G}_L$ and $\mathcal{G}_K$ and a deterministic Muller automaton $\mathcal{A}_X = (Q, \Sigma, \delta, q_0, \mathcal{G}_X)$ where $X \in \{L, K\}$ such that $L = L(\mathcal{A}_L)$ and $K = L(\mathcal{A}_K)$. Moreover, $\mathcal{G}_L \cap \mathcal{G}_K = \emptyset$, and n being even, $G_n \cap L = \emptyset$. We now construct a family of acceptance components $\mathcal{F}_i$ for $0 \leq i \leq n$. The idea is that these components will generate respectively the sequence of languages $(H_i)_{0 \leq i \leq n}$, and more importantly, they will each be cyclic closures.
First, let

$$\mathcal{F}_0 := \widehat{\mathcal{G}_K}$$

For $i \leq n$ and i odd,

$$\mathcal{F}_i := \widehat{\mathcal{F}_{i-1} \cap \mathcal{G}_L}$$

For $i \leq n$ and i even,

$$\mathcal{F}_i := \widehat{\mathcal{F}_{i-1} \cap \mathcal{G}_K}$$

We can think of this construction as starting from the essential sets containing one of $\mathcal{A}_K$'s accepting sets as a subset, and at the next step we consider essential sets containing the $F \in \mathcal{F}_0$ which are accepting for $\mathcal{A}_L$.

Let H_i be the ω-regular language over Σ which is recognized by the deterministic Muller $\mathcal{A}_i = (Q, \Sigma, \delta, q_0, \mathcal{F}_i)$. A small detail : the $n+1$ automata $\mathcal{A}_i$ and the two $\mathcal{A}_L$, $\mathcal{A}_K$ are essentially different versions of the same machine, working with the same state set Q and following the same transition fuction δ with initial state q_0, yet the decision procedure of the automaton will obviously vary depending upon the acceptance component.

We now verify the sequence $(H_i)_{0 \leq i \leq n}$ satisfies the desired properties :

1. that each language H_i is Büchi G_δ is given by Theorem 14.
2. Lemma 11.1 gives that $\mathcal{F}_{i+1} \subseteq \mathcal{F}_i$, and therefore $H_{i+1} \subseteq H_i$ for $0 \leq i < n$.
3. $K \subseteq H_0$ by definition of $\mathcal{F}_0$.
4. For $i < n$, i odd, $H_i \cap K \subseteq H_{i+1}$: Since H_i is generated by $\mathcal{F}_i = \widehat{\mathcal{F}_{i-1}} \cap \mathcal{F}_L$, the language $H_i \cap K$ is generated by $\mathcal{F}_i \cap \mathcal{F}_L \subseteq \widehat{\mathcal{F}_i} \cap \mathcal{F}_K$, hence by Lemma 11.1 again $H_i \cap K \subseteq H_{i+1}$.
5. For $i < n$, i even, similar reasoning as (4) above gives $H_i \cap L \subseteq H_{i+1}$.
6. $H_n \cap L = \emptyset$ For this to hold, it suffices to show $\mathcal{F}_n \cap \mathcal{F}_L = \emptyset$. Verifying this turns out to be the most technical part of the proof, and so we refer the reader to [1] for the details.

This shows family of Büchi G_δs $(H_i)_{0 \leq i \leq n}$ verifies conditions (1)-(4) of the statement of the theorem.

□

We can prove the following "separation" theorem.

Theorem 15. *Let K and L be two ω-regular languages over Σ such that $K \cap L = \emptyset$. Suppose there exists a decreasing sequence of G_δ subsets of Σ^ω, $G_0 \supseteq ... \supseteq G_n$ such that $\bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (G_i - G_{i+1})$ [9] separates K from L, i.e.*

$$K \subseteq \bigcup_{\substack{0 \leq i \leq n \\ i\text{even}}} (G_i - G_{i+1}) \qquad \wedge \qquad [\bigcup_{\substack{0 \leq i \leq n \\ i\text{even}}} (G_i - G_{i+1})] \cap L = \emptyset \tag{23}$$

Then there is a decreasing sequence of Büchi $G_\delta s$, $H_0 \supseteq ... \supseteq H_n$, such that $\bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (H_i - H_{i+1})$ separates K from L.

Proof. One can check that sequence$(G_i)_{0 \leq i \leq n}$ satisfies the 4 conditions of Theorem 15, hence we obtain a family of Büchi G_δs $(H_i)_{0 \leq i \leq n}$ with the same properties. Then the set $\bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (H_i - H_{i+1})$ also separates K from L. □

We are finally ready to prove (20) from Theorem 12. We restate it here for practicality.

Theorem 16 (Theorem 12, reformulation)**.** *Suppose L is an ω-regular language over Σ such that $L \in \mathcal{D}^{\mathcal{R}}_{n+1}$, i.e. $L = \bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (G_i - G_{i+1})$ for a decreasing sequence of G_δ subsets of Σ^ω.Then there is a decreasing sequence of Büchi G_δ languages over Σ $(H_i)_{0 \leq i \leq n}$ such that $L = \bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (H_i - H_{i+1})$, and thus $L \in \mathbb{D}_{n+1}$.*

Proof. The existence of the family $(H_i)_{0 \leq i \leq n}$ follows from the separation theorem, taking L and L^C as the disjoint ω-regular languages (recall the class ω-REG is closed by complementation).
Then $L = \bigcup_{\substack{0 \leq i \leq n \\ i \text{even}}} (H_i - H_{i+1})$, and so

$$L \in \mathcal{D}^{\mathcal{R}}_{n+1} \Rightarrow L \in \mathbb{D}_{n+1}$$

This completes the proof. □

9. when n is even we take $G_{n+1} = \emptyset$

4.3.1 Decidability ; Complexity of Muller automata

We end with the decidability results. As we have seen, a Muller automaton $\mathcal{A}$ accepts a G_δ ω-regular language is full table, so we associate the level $\mathcal{D}_1$ with these Muller automata. Since a language in the class $\mathcal{D}_2$ (languages L which can be written as $L = L_1 - L_2$, $L_1 \supseteq L_2$)is strictly more complex than the G_δ languages, yet still ω-regular (and thus in Δ_3^0), there exists a deterministic Muller automaton $\mathcal{B}$ accepting this language. Consequently, we can say $\mathcal{B}$ is more complex than $\mathcal{A}$– in this way Barua easily defines a hierarchy on Muller automata.
The rank of complexity of a Muller automaton in this sense of Barua is decidable : the proofs of Theorem 14 and 15 implicitly give the necessary algorithm.

In [1] these complexity classes are denoted by $\mathcal{M}_n$ and are defined on Muller automata $\mathcal{A}$ according to the rank of the language recognized in the difference hierarchy :

$$\mathcal{M}_n := \{\mathcal{A} \mid L(\mathcal{A}) \in \mathcal{D}_n\}$$

Theorem 17. *Given a deterministic Muller automaton $\mathcal{A}$ and some $n < \omega$, there is an algorithm for deciding whether $\mathcal{A} \in \mathcal{M}_n$.*

Corollary 4. *Given an ω-regular language $L \subseteq \Sigma^\omega$, it is decidable at which n L resides in the Hausdorff Kuratowski difference hierarchy for ${\Delta_3^0}^{\mathcal{R}}$.*

5 Conclusion

The purpose of this research project was to investigate compatible frameworks for determining the complexity of infinite sequences over a finite set. Specifically, we saw how the chain and superchain measures are a pure structural gauge on the class of ω-regular languages, i.e. they are invariants of the ω regular subsets of Σ^ω . It is one of the finest hierarchies one can put on ω-REG. We took Wagner's classification results and tried to relate them to the set theoretic results about classes of languages as defined by different automata.

The theme of alternation observable in Wagner's measures is continued in looking at the connection between the Hausdorff Kuratowski difference hierarchy of Δ_3^0 sets, first done by Barua in [1], the other main paper this project focused on. There, we saw a similar difference hierarchy can be defined on ω-REG. Since the ω-regular Δ_3^0 subsets do not nearly exhaust the entire class Δ_3^0, it is not immediate that we have level equivalence between Barua's hierarchy and the relativized Hausdorff-Kuratowski one, but using automata-theoretic properties of the generating class of ω-REG (the Büchi $G_\delta s$) Barua was able to extend a result done by Landweber almost 20 years prior.

One advantage of working with finite state machines was that many classification questions became decidable, and there has also been a lot of work done in automata theory analyzing the efficiency of the necessarily terminating algorithms. Thus in some cases, we can consider the theory of finite state automata on infinite languages as bridging classical results modern applications.

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